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1853.

THE

FIRST SIX BOOKS

OF

THE ELEMENTS OF EUCLID,

&c.

THE

FIRST SIX BOOKS

OF

THE ELEMENTS OF EUCLID,

WITH

A COMMENTARY AND GEOMETRICAL EXERCISES.

TO WHICH ARE ANNEXED

A TREATISE ON SOLID GEOMETRY,

AND A SHORT ESSAY

ON

THE ANCIENT GEOMETRICAL ANALYSIS.

FOR THE USE OF STUDENTS IN AND PREPARING FOR THE UNIVERSITY OF LONDON.

BY THE

REV. DIONYSIUS LARDNER, LL.D., F.R.S. L. & E.
PROFESSOR OF NATURAL PHILOSOPHY AND ASTRONOMY IN THE UNIVERSITY OF LONDON, M.R.I.A., HON. F.C.P.S., F. ASTR. SOC. LON. ETC.

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[illegible] Esq.
of
Rev. John [illegible] LL.D.
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THIS WORK

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PREFACE.

Two thousand years have now rolled away since Euclid's Elements were first used in the School of Alexandria, and to this day they continue to be esteemed the best introduction to mathematical science. They have been adopted as the basis of geometrical instruction in every part of the globe to which the light of science has penetrated; and, while in every other department of human knowledge there have been almost as many manuals as schools, in this, and in this only, one work has, by common consent, been adopted as an universal standard. Euclid has been translated into the languages of England, France, Germany, Spain, Italy, Holland, Sweden, Denmark, Russia, Egypt, Turkey, Arabia, Persia, and China. This unprecedented unanimity in the adoption of one Work as the basis of instruction, has not arisen from the absence of other Treatises on the same subject. Some of the most eminent mathematicians have written, either original Treatises, or modifications and supposed improvements of the Elements; but still the "Elements" themselves have been invariably preferred. To what can a preference so universal be attributed, if not to that singular perspicuity of arrangement, and that rigorous exactitude of demonstration, in which this celebrated Treatise has never been surpassed? 'To this,' says Playfair, 'is added every association which can render a Work venerable. It is the production of a man distinguished among the first instructors of the human race. It was almost the first ray of light which penetrated the darkness of the middle ages; and men still view with gratitude and affection the torch which rekindled the sacred fire, when it was nearly extinguished upon earth.'

It must not, however, be concealed, that, excellent as this Work is, many, whose opinions are entitled to respect, conceive that it needs much improvement; and some even think that it might be superseded with advantage by other Treatises. The Elements, as Dr. Robert Simson left them, are certainly inadequate to the purposes of instruction, in the present improved state of science. The demonstrations are characterised by prolixity, and are not always expressed in the most happy phraseology. The formalities and paraphernalia of rigour are so ostentatiously put forward, as almost to hide the reality. Endless and perplexing repetitions, which do not confer greater exactitude on the reasoning, render the demonstrations involved and obscure, and conceal from the view of the student the consecution of evidence. Independent of this defect, it is to be considered that the "Elements" contain only the naked leading truths of Geometry. Numerous inferences may be drawn, which, though not necessary as links of the great chain, and therefore subordinate in importance, are still useful, not only as exercises for the mind, but in many of the most striking physical applications. These, however, are wholly omitted by Simson, and not supplied by Playfair.

When I undertook to prepare an elementary geometrical text-book for students in, and preparing for, the University of London, I wished to render it useful in places of education generally. In this undertaking, an alternative was presented, either to produce an original Treatise on Geometry, or to modify Simson's Euclid, so as to supply all that was necessary, and to remove all that was superfluous; to elucidate what was obscure, and to abridge what was prolix; to retain geometrical rigour and real exactitude, but to reject the obtrusive and verbose display of them. The consciousness of inability to originate any work, which would bear even a remote comparison with that of the ancient Greek Geometer, would have been reason sufficient to decide upon the part I should take, were there no other considerations to direct my choice. Other considerations, however, there were, and some which seemed of great weight. The question was not, whether an elementary Treatise might not be framed superior to the "Elements," as given by Simson and Playfair; but whether an original Treatise could be produced superior to what these Elements would become, when all the improvements of which they were susceptible had been made, and when all that was found deficient had been supplied. Let us for the present admit, that a new Work were written on a plan different from that of Euclid, constructed upon different principles, built upon different data, and exhibiting the leading results of geometrical science in a different order. Let us wave also the great improbability, that even an experienced instructor should execute a Work superior to that which has been stamped with the approbation of ages, and consecrated, as it were, by the collective suffrage of the whole civilized globe. Still it may be questioned whether, on the whole, any real advantage would be gained. It is certain that all would not agree in their decision on the merits of such a Work. Euclid once superseded, every teacher would esteem his own Work the best, and every school would have its own class-book. All that rigour and exactitude, which have so long excited the admiration of men of science, would be at an end. These very words would lose all definite meaning. Every school would have a different standard; matter of assumption in one, being matter of demonstration in others, until, at length, GEOMETRY, in the ancient sense of the word, would be altogether frittered away, or be only considered as a particular application of Arithmetic and Algebra.

Independently of the disadvantages which would attend the introduction of a great number of different geometrical class-books into the schools, nearly all of which must be expected to be of a very inferior order, inconveniences of another kind would, I conceive, be produced by allowing Euclid's Elements to fall into disuse. Hitherto Euclid has been an universal standard of geometrical science. His arrangement of principles is registered in the memory of every mathematician of the present times, and is referred to in the works of every mathematician of past ages. The books of Euclid and their propositions are as familiar to the minds of all who have been engaged in scientific pursuits, as the letters of the alphabet. The same species of inconvenience, differing only in degree, would arise from disturbing this universal arrangement of geometrical principles as would be produced by changing the names and power of the letters. It is very probable, nay it is certain, that a better classification of simple sounds and articulations could be found than the commonly received vowels and consonants; yet who would advocate a change?

In expressing my sentiments respecting Euclid's Work, as compared with others which have been proposed to supersede it, I may perhaps be censured for an undue degree of confidence in a case where some respectable opinions are opposed to mine. Were I not supported in the most unqualified degree by authorities ancient and modern, the force of which seems almost irresistible, I should feel justly obnoxious to this charge. The objections which have been from time to time brought against this Work, and which are still sometimes repeated, may be reduced to two classes; those against the arrangement, and those against the reasoning. My business is not to show that Euclid is perfect either in the one respect or the other, but to show that no other elementary writer has approached so near to perfection in both. It is important to observe, that validity of reasoning and rigour of demonstration are objects which a geometer should never lose sight of, and to which arrangement and every other consideration must be subordinate. LEIBNITZ, an authority of great weight on such a subject, and not the less so as being one of the fathers of modern analysis, has declared that the geometers who have disapproved of Euclid's arrangement have vainly attempted to change it without weakening the force of the demonstrations. Their unavailing attempts, he considers to be the strongest proof of the difficulty of substituting for the chain formed by the ancient geometer, any other equally strong and valid.* WOLF also acknowledges how futile it is to attempt to arrange geometrical truths in a natural or absolutely methodical order, without either taking for granted what has not been previously established, or relaxing in a great degree the rigour of demonstration.† One of the favourite arrangements of those who object to that of Euclid, has consisted in establishing all the properties of straight lines considered without reference to their length, intersecting obliquely and at right angles, as well as the properties of parallel lines, before the more complex magnitudes called triangles are considered. In attempting this, it is curious to observe the difficulties into which these authors fall, and the expedients to which they are compelled to resort. Some find it necessary to prove that every point on a perpendicular to a given right line is equally distant from two points taken on the given right line, at equal distances from the point where the perpendicular meets it. 'They imagine,' says *Montucla*, 'that they prove this by saying that the perpendicular does not lean more to one side than the other.' Again, to prove that equal chords of a circle subtend equal arcs, they say that the uniformity of the circle produces this effect: that two circles intersect in no more than two points, and that a perpendicular is the shortest distance of a point from a right line, are propositions which they dispose of very summarily, by appealing to the evidence of the senses. They prefer an imperfect demonstration, or no demonstration at all, to any infringement of the order which they have assumed.

'There is a kind of puerility in this affectation of not mentioning a particular modification of magnitude, triangles, for example, until we have first treated of lines and angles; for if any degree of geometrical rigour be required, as many and as long demonstrations are necessary as if we had at once commenced with triangles, which, though more complex modifications of magnitude, are still so simple that the student does not require to be led by degrees to them. Some have even gone so far as to think that this affectation of a natural and absolutely methodical order contracts the mind by habituating it to a process of investigation contrary to that of discovery.'*

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* Montucla, tom. i. p. 205.

† Element. Math. tom. v. c. 3. art. 8.

The mathematicians who have attempted to improve the reasoning of Euclid, have not been more successful than those who have tried to reform his arrangement. Of the various objections which have been brought against Euclid's reasoning, two only are worthy of notice, *viz.* those respecting the twelfth axiom of the first book, which is sometimes called *Euclid's postulate*, and those which relate to his doctrine of proportion. On the former, I have enlarged so fully in Appendix II. that little remains to be said here. I have there shown that what is really assumed by Euclid, is, that 'two right lines which diverge from the same point cannot be both parallel to the same right line;' or that 'more than one parallel cannot be drawn through a given point to a given right line.' The geometers who have attempted to improve this theory, have all either committed illogicisms or assumed theorems less evident than that which has just been expressed, and which seems to me as evident as several of the other axioms. In the Appendix I have stated at length some of the theories of parallels which have been proposed to supersede that of Euclid, and have shown their defects. Numerous have been the attempts to demonstrate the twelfth axiom by the aid of the first twenty-eight propositions. Ptolemy, Proclus, Nasireddin, Clavius, Wallis, Saccheri, and a cloud of editors and commentators of former and later times, have assailed the problem without success.

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* Montucla, p. 206.

The second source of objection, on the score of reasoning, is the definition of four proportional magnitudes prefixed to the fifth book. By this definition, four magnitudes will be proportional if there be any equimultiples of the first and third which are respectively equal to equimultiples of the second and fourth. This is the common popular notion of proportion. But it is necessary to render the term more general in its geometrical application. Four magnitudes are frequently so related, that no equimultiples of the first and third are equal respectively to other equimultiples of the second and fourth, but yet have all the other properties of proportional quantities, and therefore it is necessary that they should be brought under the same definition. Euclid adapted his definition to embrace these, by declaring four magnitudes to be proportional, when every pair of equimultiples of the first and third were both greater, equal to, or less than equimultiples of the second and fourth. I agree with Playfair, in thinking that no other definition has ever been given from which the properties of proportionals can be deduced by reasonings, which, at the same time that they are perfectly rigorous, are also simple and direct. Were we content with a definition which would only include commensurate magnitudes, no difficulty would remain. But such a definition would be useless; for in almost the first instance in which it should be applied, the reasoning would either be inconclusive, or the result would not be sufficiently general.

In the second and fifth books, in addition to Euclid's demonstrations, I have in most instances given others which are rendered more clear and concise by the use of a few of the symbols of algebra, the signification of which is fully explained, and which the student will find no difficulty in comprehending. The nature of the reasoning, however, is essentially the same, the language alone in which it is expressed being different.

The commentary and deductions are distinguished from the text of the Elements, by being printed in a smaller character;

and those articles in each book which are marked thus *** the student is advised to omit until the second reading.

No part of Euclid's Elements has obtained the same celebrity, or been so universally studied as the first six books. The seventh, eighth, and ninth books treat of the Theory of Numbers, and the tenth is devoted to the Theory of Incommensurable Quantities. Instead of the eleventh and twelfth books, I have added a Treatise on Solid Geometry more suited to the present state of mathematical knowledge. For much of the materials of this treatise I am indebted to Legendre's Geometry.

Appendix I. contains a short Essay on the Ancient Geometrical Analysis, which may be read with advantage after the sixth book. The Second Appendix contains an account of the Theories of Parallels.

I have directed that the cuts of this work shall be published separately, in a small size, for the convenience of students who are taught in classes where the use of the book itself is not permitted.

33, *Percy Street, Bedford Square,*
May, 1828.

ERRATA.

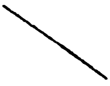
P. 65, l. 8 from the bottom, for *the product,* read *half the product.*
P. 288, l. 20 from the bottom, *dele* cylindrical,

THE
ELEMENTS OF EUCLID.

BOOK I.

DEFINITIONS.

(1) I. A *point* is that which has no parts.
(2) II. A *line* is length without breadth.
(3) III. The extremities of a line are points.
(4) IV. A *right line* is that which lies evenly between its extremities.



(5) V. A *surface* is that which has length and breadth only.
(6) VI. The extremities of a surface are lines.
(7) VII. A *plane surface* is that which lies evenly between its extremities.

(8) These definitions require some elucidation. The object of Geometry * is the properties of *figure*, and figure is defined to be the relation which subsists between the boundaries of space. Space or magnitude is of three kinds, *line*, *surface*, and *solid*. It may be here observed, once for all, that the terms used in geometrical science, are not designed to signify any real, material or physical existences. They signify certain abstracted notions or conceptions of the mind, derived, without doubt, originally from material objects by the senses, but subsequently corrected, modified, and, as it were, purified by the operations of the understanding. Thus, it is certain, that nothing exactly conformable to the geometrical notion of a right line ever existed; no edge, which the finest tool of an artist can construct, is so completely free from inequalities as to entitle it to be considered as a mathematical right line. Nevertheless, the first notion of such an edge being obtained by the senses, the process of mind by which we reject the inequalities incident upon the nicest mechanical production, and substitute for them, *mentally*, that perfect evenness which constitutes the essence of a right line, is by no means difficult. In

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* From γῆ, *terra*; and μέτρον, *mensura*.

like manner, if a pen be drawn over this paper an effect is produced, which, in common language, would be called a line, right or curved, as the case may be. This, however, cannot, in the strict geometrical sense of the term, be a *line* at all, since it has breadth as well as length; for if it had not it could not be made evident to the senses. But having first obtained this rude and incorrect notion of a line, we can imagine that, while its length remain unaltered, it may be infinitely attenuated until it ceases altogether to have breadth, and thus we obtain the exact conception of a mathematical line.

The different modes of magnitude are ideas so extremely uncompounded that their names do not admit of definition properly so called at all.* We may, however, assist the student to form correct notions of the true meaning of these terms, although we may not give rigorous logical definitions of them.

A notion being obtained by the senses of the smallest magnitude distinctly perceptible, this is called a *physical point.* If this point were indivisible even *in idea*, it would be strictly what is called a *mathematical point.* But this is not the case. No material substance can assume a magnitude so small that a smaller may not be imagined. The mind, however, having obtained the notion of an extremely minute magnitude, may proceed without limit in a mental diminution of it; and that state at which it would arrive if this diminution were infinitely continued, is a *mathematical point.*†

The introduction of the idea of motion into geometry has been objected to as being foreign to that science. Nevertheless, it seems very doubtful whether we may not derive from motion the most distinct ideas of the modes of magnitude. If a mathematical point be conceived to move in space, and to mark its course by a trace or track, that trace or track will be a *mathematical line.* As the moving point has no magnitude, so it is evident that its track can have no breadth or thickness. The places of the point at the beginning and end of its motion, are the extremities of the line, which are therefore *points.* The third of the preceding definitions is not properly a definition, but is a proposition, the truth of which may be inferred from the first two definitions.

As a *mathematical line* may be conceived to proceed from the motion of a *mathematical point,* so a *physical line* may be conceived to be generated by the motion of a *physical point.*

In the same manner as the motion of a point determines the idea of a line, the motion of a line may give the idea of a surface. If a mathematical line be conceived to move, and to leave in the space through which it passes a trace or track, this trace or track will be a surface; and since the line has no breadth, the surface can have no thickness. The initial and final position of the moving line are two boundaries or extremities of the surface, and the other extremities are the lines traced by the extreme points of the line whose motion produced the surface.

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* The name of a simple idea cannot be defined, because the several terms which compose a definition signifying several different ideas can by no means express an idea which has no manner of composition.—Locke.

† The Pythagorean definition of a point, is *'a monad having position.'*

The sixth definition is therefore liable to the same objection as the third. It is not properly a definition, but a principle, the truth of which may be derived from the fifth and preceding definitions.

It is scarcely necessary to observe, that the validity of the objection against introducing *motion* as a *principle* into the Elements of Geometry, is not here disputed, nor is it introduced as such. The preceding observations are designed merely as *illustrations* to assist the student in forming correct notions of the true mathematical significations of the different modes of magnitude. With the same view we shall continue to refer to the same mechanical idea of motion, and desire our observations always to be understood in the same sense.

The fourth definition, that of a right or straight line, is objectionable, as being unintelligible; and the same may be said of the definition (seventh) of a plane surface. Those who do not know what the words 'straight line' and 'plane surface' mean, will never collect their meaning from these definitions; and to those who *do* know the meaning of those terms, definitions are useless. The meaning of the terms 'right line' and 'plane surface' are only to be made known by an appeal to experience, and the evidence of the senses, assisted, as was before observed, by the power of the mind called *abstraction.* If a perfectly flexible string be pulled by its extremities in opposite directions, it will assume, between the two points of tension, a certain position. Were we to speak without the rigorous exactitude of geometry, we should say that it formed a *straight line.* But upon consideration, it is plain that the string has weight, and that its weight produces a flexure in it, the convexity of which will be turned towards the surface of the earth. If we conceive the weight of the string to be extremely small, that flexure will be proportionably small, and if, by the process of abstraction, we conceive the string to have no weight, the flexure will altogether disappear, and the string will be accurately a straight line.

A straight line is also sometimes defined 'to be the shortest way between two points.' This is the definition given by Archimedes, and after him by Legendre in his Geometry; but Euclid considers this as a property to be proved. In this sense, a straight line may be conceived to be that which is traced by one point moving towards another, which is quiescent.

Plato defines a straight line to be that whose extremity hides all the rest, the eye being placed in the continuation of the line.

Probably the best definition of a plane surface is, that it is such a surface that the right line, which joins every two points which can be assumed upon it, lies entirely in the surface. This definition, originally given by *Hero,* is substituted for Euclid's by R. Simson and Legendre.

Plato defined a plane surface to be one whose extremities hide all the intermediate parts, the eye being placed in its continuation.

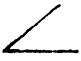
It has been also defined as 'the smallest surface which can be contained between given extremities.'

Every line which is not a straight line, or composed of straight lines, is called a *curve.* Every surface which is not a plane, or composed of planes, is called a *curved surface.*

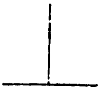
(9) VIII. A *plane angle* is the inclination of two lines to one another, in a plane, which meet together, but are not in the same direction.

This definition, which is designed to include the inclination of curves as well as right lines, is omitted in some editions of the Elements, as being useless.

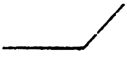
(10) IX. A *plane rectilinear angle* is the inclination of two right lines to one another, which meet together, but are not in the same right line.



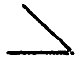
(11) X. When a right line standing on another right line makes the adjacent angles equal, each of these angles is called a *right angle*, and each of these lines is said to be *perpendicular* to the other.



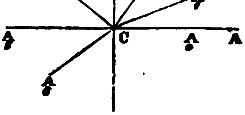
(12) XI. An *obtuse* angle is an angle greater than a right angle.



(13) XII. An *acute* angle is an angle less than a right angle.



(14) Angles might not improperly be considered as a fourth species of magnitude. Angular magnitude evidently consists of parts, and must therefore be admitted to be a species of quantity. The student must not suppose that the magnitude of an angle is affected by the length of the right lines which include it, and of whose mutual divergence it is the measure. These lines, which are called the *sides* or the *legs* of the angle, are supposed to be of indefinite length. To illustrate the nature of angular magnitude, we shall again recur to motion. Let C be supposed to be the extremity of a right line C A, extending indefinitely in the direction C A. Through the same point C, let another indefinite right line C A_0 be conceived to be drawn; and suppose this right line to revolve in the same plane round its extremity C, it being supposed at the beginning of its motion to coincide with C A. As it revolves from C A_0 to C A_1, C A_2, C A_3, &c., its divergence from C A, or, what is the same, the *angle* it makes with C A, continually increases. The line continuing to revolve, and successively assuming the positions C A_1, C A_2, C A_3, C A_4, &c., will at length coincide with the continuation C A_5 of the line C A_0 on the opposite side of the point C. When it assumes this position, it is considered by Euclid to have no inclination to C A_0,

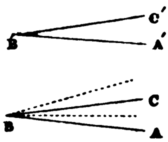


and to form no *angle* with it. Nevertheless, when the student advances further in mathematical science, he will find, that not only the line C A_6 is considered to form an angle with C A_0, but even when the revolving line continues its motion past C A_6; as for instance, to C A_8, it is still considered as forming an angle with C A_0; and this angle is measured in the direction A_0, A_5, A_4, &c. to A_0.

The point where the sides of an angle meet is called the *vertex* of the angle.

Superposition is the process by which one magnitude may be conceived to be placed upon another, so as exactly to cover it, or so that every part of each shall exactly coincide with every part of the other.

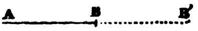
It is evident that any magnitudes which admit of superposition must be equal, or rather this may be considered as the definition of equality. Two angles are therefore equal when they admit of superposition. This may be determined thus; if the angles A B C and A′ B′ C′ are those whose equality is to be ascertained, let the vertex B′ be conceived to be placed upon the vertex B, and the side B′ A′ on the side B A, and let the remaining side B′ C′ be placed at the same side of B A with B C. If under these circumstances B′ C′ lie upon, or coincide with B C, the angles admit of superposition, and are equal, but otherwise not. If the side B′ C′ fall between B C and B A, the angle B′ is said to be *less* than the angle B, and if the side B C fall between B′ C′ and B A, the angle B′ is said to be greater than B.



As soon as the revolving line assumes such a position C A_3 that the angle A C A_3 is equal to the angle A_3 C A_6, each of those angles is called a *right angle*.

An angle is sometimes expressed simply by the letter placed at its vertex, as we have done in comparing the angles B and B′. But when the same point, as C, is the vertex of more angles than one, it is necessary to use the three letters expressing the sides as A C A_3, A_3 C A_6, the letter at the vertex being always placed in the middle.

When a line is extended, prolonged, or has its length increased, it is said to be *produced*, and the increase of length which it receives is called its *produced part*, or its *production*. Thus, if the right line A B be prolonged to A B′, it is said to be *produced through* the extremity B, and B B′ is called its *production* or *produced part*.



Two lines which meet and cross each other are said to *intersect*, and the point or points where they meet are called *points* of *intersection*. It is assumed as a self-evident truth, that two right *lines* can only intersect in one point. Curves, however, may intersect each other, or right lines, in several points.

Two right lines which intersect, or whose productions intersect, are said to be *inclined* to each other, and their inclination is measured by the angle which they include. The angle included by two right lines is sometimes called the angle *under* those lines; and right lines which include equal angles are said to be equally inclined to each other.

It may be observed, that in general when right lines or plane surfaces are spoken of in Geometry, they are considered as extended or *produced* indefinitely. Whenever a determinate portion of a right line is spoken of, it is generally called a *finite* right line. When a right line is said to be given, it is generally meant that its position or direction on a plane is given. But when a *finite* right line is given, it is understood, that not only its position, but its length is given. These distinctions are not always rigorously observed, but it never happens that any difficulty arises, as the meaning of the words is always sufficiently plain from the context.

When the direction alone of a line is given, the line is sometimes said to be *given in position,* and when the length alone is given, it is said to be *given in magnitude.*

By the inclination of two finite right lines which do not meet, is meant the angle which would be contained under these lines if produced until they intersect.

(15) XIII. A term or boundary is the extremity of any thing.

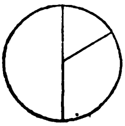
This definition might be omitted as useless.

(16) XIV. A figure is a surface, inclosed on all sides by a line or lines.

The entire length of the line or lines, which inclose a figure, is called its *perimeter.*

A figure whose surface is a plane is called a plane figure. The first six books of the Elements treat of plane figures only.

(17) XV. A *circle* is a plane figure, bounded by one continued line, called its *circumference,* or *periphery;* and having a certain point within it, from which all right lines drawn to its circumference, are equal.



If a right line of a given length revolve in the same plane round one of its extremities as a fixed point, the other extremity will describe the circumference of a circle, of which the centre is the fixed extremity.

(18) XVI. This point (from which the equal lines are drawn) is called the centre of the circle.

(19) A line drawn from the centre of a circle to its circumference is called a *radius.*

(20) XVII. A *diameter* of a circle is a right line drawn through the centre, terminated both ways in the circumference.

(21) XVIII. A semicircle is the figure contained by the diameter, and the part of the circle cut off by the diameter.

(22) From the definition of a circle, it follows immediately, that a line drawn from the centre to any point *within* the circle is less than the radius; and a line from the centre to any point *without* the circle is greater than the radius. Also, every point, whose distance from the centre is *less* than the radius, must be *within* the circle; every point whose distance from the centre is *equal* to the radius must be *on* the circle; and every point, whose distance from the centre is *greater* than the radius, is *without* the circle.

The word 'semicircle,' in Def. XVIII., assumes, that a diameter divides the circle into two equal parts. This may be easily proved by supposing the two parts, into which the circle is thus divided, placed one upon the other, so that they shall lie at the same side of their common diameter: then if the arcs of the circle which bound them do not coincide. let a radius be supposed to be drawn, intersecting them. Thus, the radius of the one will be a part of the radius of the other; and therefore, two radii of the same circle are unequal, which is contrary to the definition of a circle. (17.)

(23) XIX. A segment of a circle is a figure contained by a right line, and the part of the circumference which it cuts off.

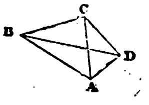
(24) XX. A figure contained by right lines only, is called a *rectilinear figure.*

The lines which include the figure are called its *sides.*

(25) XXI. A triangle is a rectilinear figure included by three sides.

A triangle is the most simple of all rectilinear figures, since less than three right lines cannot form any figure. All other rectilinear figures may be resolved into triangles by drawing right lines from any point within them to their several vertices. The triangle is therefore, in effect, the element of all rectilinear figures; and on its properties, the properties of all other rectilinear figures depend. Accordingly, the greater part of the first book is devoted to the developement of the properties of this figure.

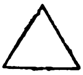
(26) XXII. A quadrilateral figure is one which is bounded by four sides. The right lines, AB, CD, connecting the vertices of the opposite angles of a quadrilateral figure, are called its *diagonals.*



(27) XXIII. A polygon is a rectilinear figure, bounded by more than four sides.

Polygons are called pentagons, hexagons, heptagons, &c., according as they are bounded by five, six, seven, or more sides. A line joining the vertices of any two angles which are not adjacent is called a diagonal of the polygon.

(28) XXIV. A triangle, whose three sides are equal, is said to be equilateral.

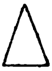


In general, all rectilinear figures whose sides are equal, may be said to be equilateral.

Two rectilinear figures, whose sides are respectively equal each to each, are said to be *mutually equilateral.* Thus, if two triangles have each sides of three, four, and five feet in length, they are *mutually equilateral*, although neither of them is *an equilateral triangle.*

In the same way a rectilinear figure having all its angles equal, is said to be *equiangular*, and two rectilinear figures whose several angles are equal each to each, are said to be *mutually equiangular.*

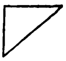
(29) XXV. A triangle which has only two sides equal is called an isosceles triangle.



The equal sides are generally called the *sides*, to distinguish them from the third side, which is called the *base.*

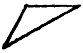
(30) XXVI. A *scalene triangle* is one which has no two sides equal.

(31) XXVII. A right-angled triangle is that which has a right angle.

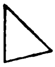


That side of a right-angled triangle which is opposite to the right angle is called the *hypotenuse.*

(32) XXVIII. An obtuse-angled triangle is that which has an obtuse angle.

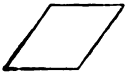


(33) XXIX. An acute-angled triangle is that which has three acute angles.

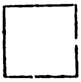


It will appear hereafter, that a triangle cannot have more than one angle right or obtuse, but may have all its angles acute.

(34) XXX. An equilateral quadrilateral figure is called a *lozenge.*

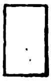


(35) XXXI. An equiangular lozenge is called a *square.*



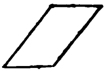
We have ventured to change the definition of a square as given in the text. A lozenge, called by Euclid a *rhombus*, when equiangular, must have all its angles right, as will appear hereafter. Euclid's definition, which is 'a lozenge all whose angles are right,' therefore, contains more than sufficient for a definition, inasmuch as, had the angles been merely *defined* to be equal, they might be *proved* to be right. To effect this change in the definition of a square, we have transposed the order of the last two definitions.

(35) XXXII. An *oblong* is a quadrilateral, whose angles are all right, but whose sides are not equal.



This term is not used in the Elements, and therefore the definition might have been omitted. The same figure is defined in the second book, and called a *rectangle*. It would appear that this circumstance of defining the same figure twice must be an oversight.

(36) XXXIII. A *rhomboid* is a quadrilateral, whose opposite sides are equal.

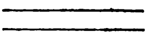


This definition and the term *rhomboid* are superseded by the term *parallelogram*, which is a quadrilateral, whose opposite sides are parallel. It will be proved hereafter, that if the opposite sides of a quadrilateral be equal, it must be a parallelogram. Hence, a distinct denomination for such a figure is useless.

(37) XXXIV. All other quadrilateral figures are called *trapeziums*.

As *quadrilateral figure* is a sufficiently concise and distinct denomination, we shall restrict the application of the term *trapezium* to those quadrilaterals which have two sides parallel.

(38) XXXV. Parallel right lines are such as are in the same plane, and which, being produced continually in both directions, would never meet.



It should be observed, that the circumstance of two right lines, which are produced indefinitely, never meeting, is not sufficient to establish their parallelism. For two right lines which are not in the same plane can never meet, and yet are not parallel. Two things are indispensably necessary to establish the parallelism of two right lines, 1°, that they be in the same plane, and 2°, that when indefinitely produced, they never meet. As in the first six books of the Elements all the lines which are considered are supposed to be in the same plane, it will be only necessary to attend to the latter criterion of parallelism.

POSTULATES.

(39) I. Let it be granted that a right line may be drawn from any one point to any other point.

(40) II. Let it be granted that a finite right line may be produced to any length in a right line.

(41) III. Let it be granted that a circle may be described with any centre at any distance from that centre.

(42) The object of the postulates is to declare, that the only instruments, the use of which is permitted in Geometry, are the *rule* and *compass*. The *rule* is an instrument which is used to direct the pen or pencil in drawing a right line; but it should be observed, that the geometrical rule is not supposed to be *divided* or *graduated*, and, consequently, it does not enable us to draw a right line of any proposed length. Neither is it permitted to place any permanent mark or marks on any part of the rule, or we should be able by it to solve the second proposition of the first book, which is *to draw from a given point a right line equal to another given right line.* This might be done by placing the rule on the given right line, and marking its extremities on the rule, then placing the mark corresponding to one extremity at the given point, and drawing the pen along the rule to the second mark. This, however, is not intended to be granted by the postulates.

The third postulate concedes the use of the compass, which is an instrument composed of two straight and equal legs united at one extremity by a joint, so constructed that the legs can be opened or closed so as to form any proposed angle. The other extremities are points, and when the legs have been opened to any degree of divergence, the extremity of one of them being fixed at a point, and the extremity of the other being moved around it in the same plane will describe a circle, since the distance between the points is supposed to remain unchanged. The fixed point is the centre; and the distance between the points, the radius of the circle.

It is not intended to be conceded by the third postulate that a circle can be described round a given centre with a radius of a given length; in other words, it is not granted that the legs of the compass can be opened until the distance between their points shall equal a given line.

AXIOMS.

(43) I. Magnitudes which are equal to the same are equal to each other.

(44) II. If equals be added to equals the sums will be equal.

(45) III. If equals be taken away from equals the remainders will be equal.

(46) IV. If equals be added to unequals the sums will be unequal.

(47) V. If equals be taken away from unequals the remainders will be unequal.

(48) VI. The doubles of the same or equal magnitudes are equal.

(49) VII. The halves of the same or equal magnitudes are equal.

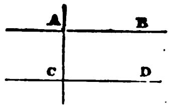
(50) VIII. Magnitudes which coincide with one another, or exactly fill the same space, are equal.

(51) IX. The whole is greater than its part.

(52) X. Two right lines cannot include a space.

(53) XI. All right angles are equal.

(54) XII. If two right lines (A B, C D) meet a third right line (A C) so as to make the two interior angles (B A C and D C A) on the same side less than two right angles, these two right lines will meet if they be produced on that side on which the angles are less than two right angles.



(55) The geometrical axioms are certain general propositions, the truth of which is taken to be self-evident, and incapable of being established by demonstration. According to the spirit of this science, the number of axioms should be as limited as possible. A proposition, however self-evident, has no title to be taken as an axiom, if its truth can be deduced from axioms already admitted. We have a remarkable instance of the rigid adherence to this principle in the twentieth proposition of the first book, where it is proved that ‘two sides of a triangle taken together are greater than the third;’ a proposition which is quite as self-evident as any of the received axioms, and much more self-evident than several of them.

On the other hand, if the truth of a proposition cannot be established by demonstration, we are compelled to take it as an axiom, *even though it be not self-evident.* Such is the case with the twelfth axiom. We shall postpone our observations on this axiom, however, for the present, and have to request that the student will omit it until he comes to read the commentary on the twenty-eighth proposition.

Two magnitudes are said to be *equal* when they are capable of exactly covering one another, or filling the same space. In the most ordinary practical cases we use this test for determining equality; we apply the two things to be compared one to the other, and immediately infer their equality from their coincidence.

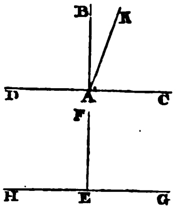
By the aid of this definition of equality we conceive that the second and third axioms might easily be deduced from the first. We shall not however pursue the discussion here.

The fourth and fifth axioms do not seem to express all that they should or might express. After equals are added to, or subtracted from unequals, the inequality is declared to remain; but it should also be declared, that the magnitude which was the greater before the addition or subtraction continues still to be the greater, and that its excess over the lesser magnitude has suffered no change. This might be expressed thus, 'the difference of two unequal quantities is not changed by the addition or subtraction of equal quantities to or from both.'

It is not difficult to deduce the sixth and seventh axioms from the preceding ones.

The eighth axiom is more properly a definition of the meaning of equal magnitudes.

The tenth axiom may be presented under various forms. It is equivalent to stating, that only one right line can be drawn between two given points. For if two different right lines could be drawn from one given point to another, these two right lines would evidently enclose a space between them. It is also equivalent to stating, that two right lines indefinitely produced cannot intersect each other in more than one point; for if they intersected at a second point, the parts between the two points would enclose a space. It is also equivalent to stating that a rectilinear figure cannot have less than three sides.



The eleventh axiom evidently admits of proof. Let A B and E F be perpendicular to D C and H G; the right angles B A C and B A D are equal by (11), as also the right angles F E G and F E H. But to prove that the right angle F E G is equal to the right angle B A C, let the point E be conceived to be placed upon A, and the line H E G on D A C, the perpendiculars E F and A B lying at the same side of the coincident lines, these perpendicular will coincide, and therefore the angle F E G is equal to B A C. For if E F do not coincide with A B let it take any other position as A K. Since A K is perpendicular to D C, the angle K A C is equal to the angle K A D, and therefore K A C is greater than the angle B A D; but since B A is perpendicular to A D the angle B A D is equal to B A C, therefore K A C is greater than B A C, that is, a part is greater than the whole, which is absurd. Therefore E F must coincide with A B.

The postulates may be considered as axioms. The first postulate, which declares the possibility of one right line joining two given points, is as much an axiom as the tenth axiom, which declares the impossibility of more than one right line joining them.

In like manner, the second postulate which grants the power of producing a line may be considered as an axiom, declaring that every finite straight line may have another placed at its extremity so as to form with it one continued straight line. In fact, the straight line thus placed will be its production. This postulate is assumed as an axiom in the fourteenth proposition of the first book.

(56) Those results which are obtained in geometry by a process of reasoning are called *propositions.* Geometrical propositions are of two species, *problems* and *theorems.*

A *problem* is a proposition in which something is proposed to be done; as a line to be drawn under some given conditions, some figure to be constructed, &c. The solution of the problem consists in showing how the thing required may be done by the aid of the rule and compass. The demonstration consists in proving that the process indicated in the solution really attains the required end.

A *theorem* is a proposition in which the truth of some principle is asserted. The object of the demonstration is to show how the truth of the proposed principle may be deduced from the axioms and definitions or other truths previously and independently established.

A problem is analogous to a postulate, and a theorem to an axiom.

A postulate is a problem, the solution of which is assumed.

An axiom is a theorem, the truth of which is granted without demonstration.

In order to effect the demonstration of a proposition, it frequently happens that other lines must be drawn besides those which are actually engaged in the enunciation of the proposition itself. The drawing of such lines is generally called the *construction.*

A *corollary* is an inference deduced immediately from a proposition.

A *scholium* is a note or observation on a proposition not containing any inference, or, at least, none of sufficient importance to entitle it to the name of a *corollary.*

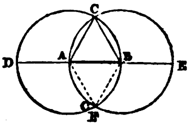
A *lemma* is a proposition introduced for the purpose of establishing some more important proposition.

PROPOSITION I. PROBLEM.

(57) On a given finite right line (A B) to construct an equilateral triangle.

Solution.

With the centre A and the radius A B let a circle B C D be described (41), and with the centre B and the radius B A let another circle A C E be described. From a point of intersection C of these circles let right lines be drawn to the extremities A and B of the given right line (39). The triangle A C B will be that which is required.



Demonstration.

It is evident that the triangle A C B is constructed on the given right line A B. But it is also equilateral; for the lines A C and A B, being radii of the same circle B C D, are equal (17), and also B C and B A, being radii of the same circle A C E, are equal. Hence the lines B C and A C, being equal to the same line A B, are equal to each other (43). The three sides of the triangle A B C are therefore equal, and it is an equilateral triangle (28).

(58) In the solution of this problem it is assumed that the two circles intersect, inasmuch as the vertex of the equilateral triangle is a point of intersection. This, however, is sufficiently evident if it be considered that a circle is a continued line which includes space, and that in the present instance each circle passing through the centre of the other must have a part of its circumference within that other, and a part without it, and must therefore intersect it.

It follows from the solution, that as many different equilateral triangles can be constructed on the same right line as there are points in which the two circles intersect. It will hereafter be proved that two circles cannot intersect in more than two points, but for the present this may be taken for granted.

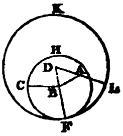
Since there are but two points of intersection of the circles, there can be but two equilateral triangles constructed on the same finite right line, and these are placed on opposite sides of it, their vertices being at the points C and F.

After having read the first book of the elements, the student will find no difficulty in proving that the triangles C F E and C D F are equilateral. These lines are not in the diagram, but may easily be supplied.

PROPOSITION II. PROBLEM.

(59) From a given point (A) to draw a right line equal to a given finite right line (B C).

Solution.



Let a right line be drawn from the given point A to either extremity B of the given finite right line B C (39). On the line A B let an equilateral triangle A D B be constructed (I). With the centre B and the radius B C let a circle be described (41). Let D B be produced to meet the circumference of this circle in F (40), and with the centre D and the radius D F let another circle F L K be described. Let the line D A be produced to meet the circumference of this circle in L. The line A L is then the required line.

Demonstration.

The lines D L and D F are equal, being radii of the same circle F L K (17). Also the lines D A and D B are equal, being sides of the equilateral triangle B D A. Taking the latter from the former, the remainders A L and B F are equal (45). But B F and B C are equal, being radii of the same circle F C H (17), and since A L and B C are both equal to B F, they are equal to each other (43). Hence A L is equal to B C, and is drawn from the given point A, and therefore solves the problem.

⁂ The different positions which the given right line and given point may have with respect to each other, are apt to occasion such changes in the diagram as to lead the student into error in the execution of the construction for the solution of this problem.

Hence it is necessary that in solving this problem the student should be guided by certain *general* directions, which are independent of any particular arrangement which the several lines concerned in the solution may assume. If the student is governed by the following general directions, no change which the diagram can undergo will mislead him.

1° The given point is to be joined with *either* extremity of the given right line. (Let us call the extremity with which it is connected, the *connected extremity* of the given right line; and the line so connecting them, the *joining line.*)

2° The centre of the first circle is the *connected extremity* of the given right line; and its radius, the given right line.

3° The equilateral triangle mav be constructed on *either side* of the joining line.

4° The side of the equilateral triangle which is produced to meet the circle, is that side which is opposite to the given point, and it is produced through the centre of the first circle till it meets its circumference.

⁂ Articles thus marked may be omitted by the student in the first reading.

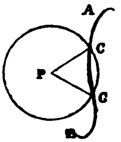
5° The centre of the second circle is that vertex of the triangle which is opposite to the joining line, and its radius is made up of that side of the triangle which is opposite to the given point, and its production which is the radius of the first circle. So that the radius of the second circle is the sum of the side of the triangle and the radius of the first circle.

6° The side of the equilateral triangle which is produced through the given point to meet the second circle, is that side which is opposite to the connected extremity of the given right line, and the production of this side is the line which solves the problem; for the sum of this line and the side of the triangle is the radius of the second circle, but also the sum of the given right line (which is the radius of the first circle) and the side of the triangle is equal to the radius of the second circle. The side of the triangle being taken away the remainders are equal.

As the given point may be joined with either extremity, there may be two different joining lines, and as the triangle may be constructed on either side of each of these, there may be four different triangles; so the right line and point being given, there are four different constructions by which the problem may be solved.

If the student inquires further, he will perceive that the solution may be effected also by producing the side of the triangle opposite the given point, not through the extremity of the right line but through the vertex of the triangle. The various consequences of this variety in the construction we leave to the student to trace.

(60) By the second proposition a right line of a given length can be inflected from a given point P upon any given line A B. For from the point P draw a right line of the given length (II), and with P as centre, and that line as radius, describe a circle. A line drawn from P to any point C, where this circle meets the given line A B, will solve the problem.



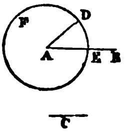
By this proposition the first may be generalized; for an *isosceles* triangle may be constructed on a given line as base, and having its side of a given length. The construction will remain unaltered, except that the radius of each of the circles will be equal to the length of the side of the proposed triangle. If this length be not greater than half the base, the two circles will not intersect, and no triangle can be constructed, as will appear hereafter.

Proposition III. Problem.

(61) From the greater (A B), of two given right lines to cut off a part equal to the less (C).

Solution.

From either extremity A of the greater let a right line A D be drawn equal to the less C (II), and with the point A as centre, and the radius A D let a circle be described (41). The part A E of the greater cut off by this circle will be equal to the less C.



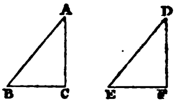
Demonstration.

For A E and A D are equal, being radii of the same circle (17); and C and A D are equal by the construction. Hence A E and C are equal.

By a similar construction, the less might be produced until it equal the greater. From an extremity of the less let a line equal to the greater be drawn, and a circle be described with this line as radius. Let the less be produced to meet this circle.

PROPOSITION IV. THEOREM.

(62) **If two triangles (B A C and E D F) have two sides (B A and A C) in the one respectively equal to two sides (E D and D F) in the other, and the angles (A and D) included by those sides also equal; the bases or remaining sides (B C and E F) will be equal, also the angles (B and C) at the base of the one will be respectively equal to those (E and F) at the base of the other which are opposed to the equal sides (i. e. B to E and C to F).**



Let the two triangles be conceived to be so placed that the vertex of one of the equal angles D shall fall upon that of the other A, that one of the sides D E containing the given equal angles shall fall upon the side A B in the other triangle to which it is equal, and that the remaining pair of equal sides A C and D F shall lie at the same side of those A B and D E which coincide.

Since then the vertices A and D coincide, and also the equal sides A B and D E, the points B and E must coincide. (If they did not the sides A B and D E would not be equal.) Also, since the side D E falls on A B, and the sides A C and D F are at the same side of A B, and the angles A and D are equal, the side D F must fall upon A C; (for otherwise the angles A and D would not be equal.)

Since the side D F falls on A C, and they are equal, the extremity F must fall on C. Since the extremities of the bases B C and E F coincide, these lines themselves must coincide, for if they did not they would include a space (52)· Hence the sides B C and E F are equal (50).

Also, since the sides E D and E F coincide respectively with B A and B C, the angles E and B are equal (50), and for a similar reason the angles F and C are equal.

Since the three sides of the one triangle coincide respectively with the three sides of the other, the triangles themselves coincide, and are therefore equal (50).

In the demonstration of this proposition, the converse of the eighth axiom (50) is assumed. The axiom states, that 'if two magnitudes coincide they must be equal.' In the proposition it is assumed, that if they be equal they must under certain circumstances coincide. For when the point D is placed on A, and the side D E on A B, it is assumed that the point E must fall upon B, because A B and D E are equal. This may, however, be proved by the combination of the eighth and ninth axioms; for if the point E did not fall upon B, but fell either above or below it, we should have either E D equal to a part of B A, or B A equal to a part of E D. In either case the ninth axiom would be contradicted, as we should have the whole equal to its part.

The same principle may be applied in proving that the side D F will fall upon A C, which is assumed in Euclid's proof.

In the superposition of the triangles in this proposition, three things are to be attended to:

1° The vertices of the equal angles are to be placed one on the other.

2° Two equal sides to be placed one on the other.

3° The other two equal sides are to be placed on the same side of those which are laid one upon the other.

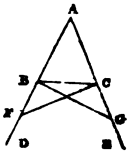
From this arrangement the coincidence of the triangles is inferred.

It should be observed, that this superposition is not assumed to be actually effected, for that would require other postulates besides the three already stated; but it is sufficient for the validity of the reasoning, if it be conceived to be possible that the triangles might be so placed. By the same principle of superposition, the following theorem may be easily demonstrated, 'If two triangles have two angles in one respectively equal to two angles in the other, and the sides lying between those angles also equal, the remaining sides and angles will be equal, and also the triangles themselves will be equal.' See prop. xxvi.

This being the first *theorem* in the Elements, it is necessarily deduced exclusively from the axioms, as the first problem must be from the postulates. Subsequent theorems and problems will be deduced from those previously established.

PROPOSITION V. THEOREM.

(63) The angles (B, C) opposed to the equal sides (A C and A B) of an isosceles triangle are equal, and if the equal sides be produced through the extremities (B and C) of the third side, the angles (D B C and E C B) formed by their produced parts and the third side are equal.



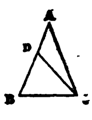
Let the equal sides A B and A C be produced through the extremities B, C of the third side, and in the produced part B D of either let any point F be assumed, and from the other let A G be cut off equal to A F (III). Let the points F and G so taken on the produced sides be connected by right lines F C and B G with the alternate extremities of the third side of the triangle.

In the triangles F A C and G A B the sides F A and A C are respectively equal to G A and A B, and the included angle A is common to both triangles. Hence (IV), the line F C is equal to B G, the angle A F C to the angle A G B, and the angle A C F to the angle A B G. If from the equal lines A F and A G, the equal sides A B and A C be taken, the remainders B F and C G will be equal. Hence, in the triangles B F C and C G B, the sides B F and F C are respectively equal to C G and G B, and the angles F and G included by those sides are also equal. Hence (IV), the angles F B C and G C B, which are those included by the third side B C and the productions of the equal sides A B and A C, are equal. Also, the angles F C B and G B C are equal. If these equals be taken from the angles F C A and G B A, before proved equal, the remainders, which are the angles A B C and A C B opposed to the equal sides, will be equal.

(64) Cor.—Hence, in an equilateral triangle the three angles are equal; for by this proposition the angles opposed to every two equal sides are equal.

Proposition VI. Theorem.

(65) If two angles (B and C) of a triangle (B A C) be equal, the sides (A C and A B) opposed to them are also equal.



For if the sides be not equal, let one of them A B be greater than the other, and from it cut off D B equal to A C (III), and draw C D.

Then in the triangles D B C and A C B, the sides D B and B C are equal to the sides A C and C B respectively, and the angles D B C and A C B are also equal; therefore (IV) the triangles themselves D B C and A C B are equal, a part equal to the whole, which is absurd; therefore neither of the sides A B or A C is greater than the other; they are therefore equal to one another.

(66) Cor.—Hence every equiangular triangle is also equilateral, for the sides opposed to every two equal angles are equal.

In the construction for this proposition it is necessary that the part of the greater side which is cut off equal to the less, should be measured upon the greater side B A from vertex (B) of the equal angle, for otherwise the fourth proposition could not be applied to prove the equality of the part with the whole.

It may be observed generally, that when a part of one line is cut off equal to another, it should be distinctly specified from which extremity the part is to be cut.

This proposition is what is called by logicians the *converse* of the fifth. It cannot however be inferred from it by the logical operation called *conversion;* because, by the established principles of the Aristotelian logic, *an universal affirmative admits no simple converse.* This observation applies generally to those propositions in the Elements which are converses of preceding ones.

The demonstration of the sixth is the first instance of indirect proof which occurs in the Elements. The force of this species of demonstration consists in showing that a principle is true, because some manifest absurdity would follow from supposing it to be false.

This kind of proof is considered inferior to *direct* demonstration, because it only proves that a thing *must* be so, but fails in showing why it should be so; whereas *direct* proof not only shows that the thing is so, but *why* it is so. Consequently, indirect demonstration is never used, except where no direct proof can be had. It is used generally in proving principles which are nearly self-evident, and in the Elements is oftenest used in establishing the *converse* propositions. Examples will be seen in the 14th, 19th, 25th, and 40th propositions of this book

PROPOSITION VII. THEOREM.

(67) On the same right line (A B), and on the same side of it, there cannot be constructed two triangles, (A C B, A D B) whose conterminous sides (A C and A D, B C and B D) are equal.

If it be possible, let the two triangles be constructed, and,

First,—Let the vertex of each of the triangles be without the other triangle, and draw C D.

Because the sides A D and A C of the triangle C A D are equal (hyp.)*, the angles A C D and A D C are equal (V); but

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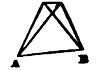
* The *hypothesis* means the *supposition;* that is, the part of the enunciation of the proposition in which something is supposed to be granted true, and from which the proposed conclusion is to be inferred. Thus in the seventh proposition the hypothesis is, that the triangles stand on the same side of their base, and that their conterminous sides are equal, and the conclusion is a manifest absurdity, which proves that the hypothesis must be false.

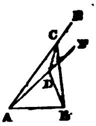
In the fourth proposition the hypothesis is, that two sides and the included angle of one triangle are respectively equal to two sides and the included angle of the other; and the conclusion deduced from this hypothesis is, that the remaining side and angles in the one triangle are respectively equal to the remaining side and angles in the other triangle.

A C D is greater than B C D (51), therefore A D C is greater than B C D; but the angle B D C is greater than A D C (51), and therefore B D C is greater than B C D; but in the triangle C B D, the sides B C and B D are equal (hyp.), therefore the angles B D C and B C D are equal (V); but the angle B D C has been proved to be greater than B C D, which is absurd: therefore the triangles constructed upon the same right line cannot have their conterminous sides equal, when the vertex of each of the triangles is without the other.



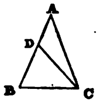
Secondly,—Let the vertex D of one triangle be within the other; produce the sides A C and A D, and join C D.

Because the sides A C and A D of the triangle C A D are equal (hyp.), the angles E C D and F D C are equal (V); but the angle B D C is greater than F D C (51), therefore greater than E C D; but E C D is greater than B C D (51), and therefore B D C is greater than B C D; but in the triangle C B D, the sides B C and B D are equal (hyp.), therefore the angles B D C and B C D are equal (V); but the angle B D C has been proved to be greater than B C D, which is absurd: therefore triangles constructed upon the same right line cannot have their conterminous sides equal, if the vertex of one of them is within the other.



Thirdly,—Let the vertex D of one triangle be on the side A B of the other, and it is evident that the sides A B and B D are not equal.

Therefore in no case can two triangles, whose conterminous sides are equal, be constructed at the same side of the given line.

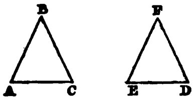


This proposition seems to have been introduced into the Elements merely for the purpose of establishing that which follows it. The demonstration is that form of argument which logicians call a *dilemma,* and is the only case in which this species of argument occurs in the Elements. If two triangles whose conterminous sides are equal could stand on the same side of the same base, the vertex of the one must necessarily either fall within the other or without it, or on one of the sides of it; accordingly, it is successively proved in the demonstration, that to suppose it in any of these positions would lead to a contradiction in terms. It is not supposed that the vertex of the one could fall on the vertex of the other; for that would be supposing the two triangles to be one and the same, whereas they are, by hypothesis, different.

In the Greek text there is but one (the first) of the cases of this proposition given. It is however conjectured, that the second case must have been formerly in the text, because it is the only instance in which Euclid uses that part of the fifth proposition which proves the equality of the angles below the base. It is argued, that there must have been some reason for introducing into the fifth a principle which follows at once from the thirteenth; and that none can be assigned except the necessity of the principle in the second case of the seventh. The third case required to be mentioned only to preserve the complete logical form of the argument.

PROPOSITION VIII. THEOREM.

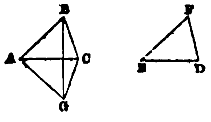
(68) If two triangles (A B C and E F D) have two sides of the one respectively equal to two sides of the other (A B to E F and B C to F D), and also have the base (A C) equal to the base (E D), then the angles (B and F) contained by the equal sides are equal.



For if the equal bases A C, E D be conceived to be placed one upon the other, so that the triangles shall lie at the same side of them, and that the equal sides A B and E F, C B and D F be conterminous, the vertex B must fall on the vertex F; for to suppose them not coincident would contradict the seventh proposition. The sides B A and B C being therefore coincident with F E and E D, the angles B and F are equal.

(69) It is evident that in this case all the angles and sides of the triangles are respectively equal each to each, and that the triangles themselves are equal. This appears immediately by the eighth axiom.

In order to remove from the threshold of the Elements a proposition so useless, and, to the younger students, so embarrassing as the seventh, it would be desirable that the eighth should be established independently of it. There are several ways in which this might be effected. The following proof seems to be liable to no objection, and establishes the eighth by the fifth.

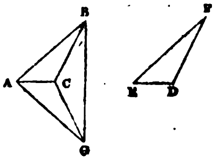


Let the two equal bases be so applied one upon the other that the equal sides shall be conterminous, and that the triangles shall lie at opposite sides of them, and let a right line be conceived to be drawn joining the vertices.

1° Let this line intersect the base.

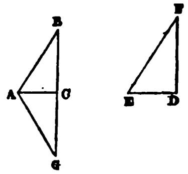
Let the vertex F fall at G, the side E F in the position A G, and D F in the position C G. Hence B A and A G being equal, the angles G B A and B G A are equal (V). Also C B and C G being equal, the angles C G B and C B G are equal (V). Adding these equals to the former, the angles A B C and A G C are equal; that is, the angles E F D and A B C are equal.

2° Let the line G B fall outside the coincident bases, and let A C be produced to meet it.



The angles G B A and B G A, and also B G C and G B C are proved equal as before; and taking the latter from the former, the remainders, which are the angles A G C and A B C, are equal, but A G C is the angle F.

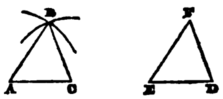
3° Let the line A G pass through either extremity of the base.



In this case it follows immediately (V) that the angles A B C and A G C are equal; for the lines B C and C G must coincide with B G, since each has two points upon it (52).

Hence in every case the angles B and F are equal.

This proposition is also sometimes demonstrated as follows.



Conceive the triangle E F D to be applied to A B C, as in Euclid's proof. Then, because E F is equal to A B, the point F must be in the circumference of a circle described with A as centre, and A B as radius. And for the same reason, F must be on a circumference with the centre C, and the radius C B. The vertex must therefore be at the point where these circles meet. But the vertex B must be also at that point; wherefore, &c.

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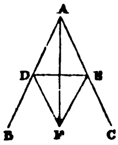
Proposition IX. Problem.

(70) To bisect a given rectilinear angle (B A C).

Solution.



Take any point D in the side A B, and from A C cut off A E equal to A D (III), draw D E, and upon it describe an equilateral triangle D F E (I) at the side remote from A. The right line joining the points A and F bisects the given angle B A C.

Demonstration.

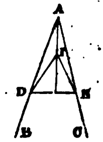
Because the sides A D and A E are equal (const.), and the side A F is common to the triangles F A D and F A E, and the base F D is also equal to F E (const.); the angles D A F and E A F are equal (VIII), and therefore the right line A F bisects the given angle.

By this proposition an angle may be divided into 4, 8, 16, &c. equal parts, or, in general, into any number of equal parts which is expressed by a power of *two*.

It is necessary that the equilateral triangle be constructed on a different side of the joining line D E from that on which the given angle is placed, lest the vertex F of the equilateral triangle should happen to coincide with the vertex A of the given angle; in which case there would be no joining line F A, and therefore no solution.

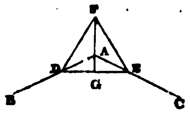
In these cases, however, in which the vertex of the equilateral triangle does not coincide with that of the given angle, the problem can be solved by constructing the equilateral triangle on the same side of the joining line D E with the given angle. Separate demonstrations are necessary for the two positions which the vertices may assume.

1. Let the vertex of the equilateral triangle fall within that of the given angle.



The demonstration already given will apply to this without any modification.

2. Let the vertex of the given angle fall within the equilateral triangle.



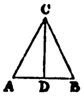
The line F A produced will in this case bisect the angle; for the three sides of the triangle D F A are respectively equal to those of the triangle E F A. Hence the angles D F A and E F A are equal (VIII). Also, in the triangles D F G and E F G the sides D F and E F are equal, the side G F is common, and the angles D F G and E F G are equal; hence (IV) the bases D G and E G are equal, and also the angles D G A and E G A. Again, in the triangles D G A and E G A the sides D G and E G are equal, A G is common, and the angles at G are equal; hence (IV) the angles D A G and E A G are equal, and therefore the angle B A C is bisected by A G.

It is evident, that an isosceles triangle constructed on the joining line D E would equally answer the purpose of the solution.

Proposition X. Problem.

(71) To bisect a given finite right line (A B).

Solution.



Upon the given line A B describe an equilateral triangle A C B (I), bisect the angle A C B by the right line C D (IX); this line bisects the given line in the point D.

Demonstration.

Because the sides A C and C B are equal (const.), and C D common to the triangles A C D and B C D, and the angles A C D and B C D also equal (const.); therefore (IV) the bases A D and D B are equal, and the right line A B is bisected in the point D.

In this and the following proposition an isosceles triangle would answer the purposes of the solution equally with an equilateral. In fact, in the demonstrations the triangle is contemplated merely as isosceles: for nothing is inferred from the equality of the base with the sides.

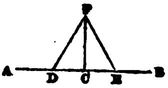
Proposition XI. Problem.

(72) From a given point (C) in a given right line (A B) to draw a perpendicular to the given line.

Solution.

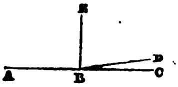
In the given line take any point D and make C E equal to C D (III), upon D E describe an equilateral triangle D F E (I), draw F C, and it is perpendicular to the given line.

Demonstration.



Because the sides D F and D C are equal to the sides E F and E C (const.), and C F is common to the triangles D F C and E F C, therefore (VIII) the angles opposite to the equal sides D F and E F are equal, and therefore F C is perpendicular to the given right line A B at the point C.

Cor.—By help of this problem it may be demonstrated, that two straight lines cannot have a common segment.



If it be possible, let the two straight lines A B C, A B D have the segment A B common to both of them. From the point B draw B E at right angles to A B; and because A B C is a straight line, the angle C B E is equal to the angle E B A; in the same manner, because A B D is a straight line, the angle D B E is equal to the angle E B A; wherefore the angle D B E is equal to the angle C B E, the less to the greater, which is impossible; therefore two straight lines cannot have a common segment.

If the given point be at the extremity of the given right line, it must be produced, in order to draw the perpendicular by this construction.

In a succeeding article, the student will find a method of drawing a perpendicular through the extremity of a line *without producing it.*

The corollary to this proposition is useless, and is omitted in some editions.

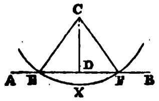
It is equivalent to proving that a right line cannot be produced through its extremity in more than one direction, or that it has but one production.

Proposition XII. Problem.

(73) To draw a perpendicular to a given indefinite right line (A B) from a point (C) given without it.

Solution.

Take any point X on the other side of the given line, and from the centre C with the radius C X describe a circle cutting the given line in E and F. Bisect E F in D (X), and draw from the given point to the point of bisection the right line C D; this line is the required perpendicular.



Demonstration.

For draw C E and C F, and in the triangles E D C and F D C the sides E C and F C, and E D and F D, are equal, (const.) and C D common; therefore (VIII) the angles E D C and F D C opposite to the equal sides E C and F C are equal, and therefore D C is perpendicular to the line A B (11).

In this proposition it is necessary that the right line A B be indefinite in length, for otherwise it might happen that the circle described with the centre C and the radius C X might not intersect it in two points, which is essential to the solution of the problem.

It is assumed in the solution of this problem, that the circle will intersect the right line in two points. The centre of the circle being on one side of the given right line, and a part of the circumference (X) on the other, it is not difficult to perceive that a part of the circumference must be also on the same side of the given line with the centre, and since the circle is a continued line it must cross the right line twice. The properties of the circle form the subject of the third book, and those which are assumed here will be established in that part of the Elements.

The following questions will afford the student useful exercise in the application of the geometrical principles which have been established in the last twelve propositions.

(74) *In an isosceles triangle the right line which bisects the vertical angle also bisects the base, and is perpendicular to the base.*

For in the two triangles into which it divides the isosceles, there are two sides (those of the isosceles) equal, and a side (the bisector) common, and the angles included by these sides equal, being the parts of the bisected angle; hence (IV) the remaining sides and angles are respectively equal; that is, the parts into which the base is divided by the bisector are equal, and the angles which the bisector makes with the base are equal. Therefore it bisects the base, and is perpendicular to it.

It is clear that the isosceles triangle itself is bisected by the bisector of its vertical angle, since the two triangles are equal.

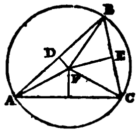
(75) It follows also, that *in an isosceles triangle the line which is drawn from the vertex to the middle point of the base bisects the vertical angle, and is perpendicular to the base.*

For in this case the triangle is divided into two triangles, which have their three sides respectively equal each to each, and the property is established by (VIII).

(76) *If in a triangle the perpendicular from the vertex on the base bisect the base, the triangle is isosceles.*

For in this case in the two triangles into which the whole is divided by the perpendicular, there are two sides (the parts of the base) equal, one side (the perpendicular) common, and the included angles equal, being right. Hence (IV) the sides of the triangle are equal.

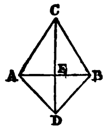
(77) *To find a point which is equidistant from the three vertical points of a triangle* (A B C).



Bisect the sides A B and B C at D and E (X), and through the points D and E draw perpendiculars, and produce them until they meet at F. The point F is at equal distances from A, B and C.

For draw F A, F B, F C. B F A is isosceles by (76), and for the same reason B F C is isosceles. Hence it is evident that F A, F C, and F B are equal.

(78) Cor.—Hence F is the centre, and F A the radius of a circle circumscribed about the triangle.



(79) *In a quadrilateral formed by two isosceles triangles* A C B *and* A D B *constructed on different sides of the same base, the diagonals intersect at right angles, and that which is the common base of the isosceles triangles is bisected by the other.*

For in the triangles C A D and C B D the three sides are equal each to each, and therefore (VIII) the angles A C E and B C E are equal. The truth of the proposition therefore follows from (74).

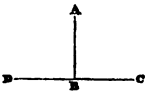
(80) Hence it follows that *the diagonal of a lozenge bisect each other at right angles.*

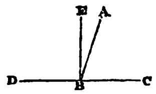
(81) It follows from (76) that *if the diagonals of a quadrilateral bisect each other at right angles it is a lozenge.*

Proposition XIII. Theorem.

(82) When a right line (A B) standing upon another (D C) makes angles with it, they are either two right angles, or together equal to two right angles.

If the right line A B is perpendicular to D C, the angles A B C and A B D are right (11). If not, draw B E perpendicular to D C (XI), and it is evident that the angles C B A and A B D together are equal to the angles C B E and E B D, and therefore to two right angles.





The words 'makes angles with it,' are introduced to exclude the case in which the line A B is at the extremity of D C.

(83) From this proposition it appears, that if several right lines stand on the same right line at the same point, and make angles with it, all the angles taken together are equal to two right angles.

Also if two right lines intersecting one another make angles, these angles taken together are equal to four right angles.

The lines which bisect the adjacent angles A B C and A B D are at right angles; for the angle under these lines is evidently half the sum of the angles A B C and A B D.

If several right lines diverge from the same point, the angles into which they divide the surrounding space are together equal to four right angles.

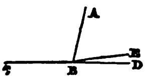
(84) When two angles as A B C and A B D are together equal to two right angles, they are said to be *supplemental*, and one is called the *supplement* of the other.

(85) If two angles as C B A and E B A are together equal to a right angle, they are said to be *complemental*, and one is said to be the *complement* of the other.

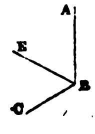
PROPOSITION XIV. THEOREM.

(86) If two right lines (C B and B D) meeting another right line (A B) at the same point (B), and at opposite sides, make angles with it which are together equal to two right angles, those right lines (C B and B D) form one continued right line.

For if possible, let B E and not B D be the continuation of the right line C B, then the angles C B A and A B E are equal to two right angles (XIII), but C B A and A B D are also equal to two right angles, by hypothesis, therefore C B A and A B D taken together are equal to C B A and A B E; take away from these equal quantities C B A which is common to both, and A B E shall be equal to A B D, a part to the whole, which is absurd, therefore B E is not the continuation of C B, and in the same manner it can be proved, that no other line except B D is the continuation of it, therefore B D forms with B C one continued right line.

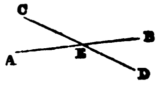


In the enunciation of this proposition, the student should be cautious not to overlook the condition that the two right lines C B and B D forming angles, which are together equal to two right angles, with B A, lie *at opposite sides* of B A. They might form angles together equal to two right angles with B A, and yet not lie in the same continued line, if as in this figure they lay at *the same side of it*. It is assumed in this proposition that the line C B has a production. This is however granted by Postulate 2.



PROPOSITION XV. THEOREM.

(87) If two right lines (A B and C D) intersect one another, the vertical angles are equal (C E A to B E D, and C E B to A E D).



Because the right line C E stands upon the right line A B, the angle A E C together with the angle C E B is equal to two right angles (XIII); and because the right line B E stands upon the right line C D, the angle C E B together with the angle B E D is equal to two right angles (XIII); therefore A E C and C E B together are equal to C E B and B E D; take away the common angle C E B, and the remaining angle A E C is equal to B E D.

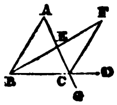
This proof may shortly be expressed by saying, that opposite angles are equal, because they have a common *supplement* (84).

It is evident that angles which have a common supplement or complement (85) are equal, and that if they be equal, their supplements and complements must also be equal.

(88) The *converse* of this proposition may easily be proved, scil. If four lines meet at a point, and the angles vertically opposite be equal, each alternate pair of lines will be in the same right line. For if C E A be equal to B E D, and also C E B to A E D, it follows, that C E A and C E B together are equal to B E D and A E D together. But all the four are together equal to four right angles (83), and therefore C E A and C E B are together equal to two right angles, therefore (XIV) A E and E B are in one continued line. In like manner it may be proved, that C E and D E are in one line.

PROPOSITION XVI. THEOREM.

(89) If one side (B C) of a triangle (B A C) be produced, the external angle (A C D) is greater than either of the internal opposite angles (A or B.)



For bisect the side A C in E (X), draw B E and produce it until E F be equal to B E (III), and join F C.

The triangles C E F and A E B have the sides C E and E F equal to the sides A E and E B (const.), and the angle C E F equal to A E B (XV), therefore the angles E C F and A are equal (IV), and therefore A C D is greater than A. In like manner it can be shown, that if A C be produced, the external angle B C G is greater than the angle B, and therefore that the angle A C D, which is equal to B C G (XV), is greater than the angle B.

(90) Cor. 1.—Hence it follows, that each angle of a triangle is less than the supplement of either of the other angles (84). For the external angle is the supplement of the adjacent internal angle (XIII).

(91) Cor. 2.—If one angle of a triangle be right or obtuse, the others must be acute. For the supplement of a right or obtuse angle is right or acute (84), and each of the other angles must be less than this supplement, and must therefore be acute.

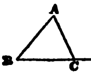
(92) Cor. 3.—More than one perpendicular cannot be drawn from the same point to the same right line. For if two lines be supposed to be drawn, one of which is perpendicular, they will form a triangle having one right angle. The other angles must therefore be acute (91), and therefore the other line is not perpendicular.

(93) Cor. 4.—If from any point a right line be drawn to a given right line, making with it an acute and obtuse angle, and from the same point a perpendicular be drawn, the perpendicular must fall at the side of the acute angle. For otherwise a triangle would be formed having a right and an obtuse angle, which cannot be (91).

(94) Cor. 5.—The equal angles of an isosceles triangle must be both acute.

Proposition XVII. Theorem.

(95) Any two angles of a triangle (B A C) are together less than two right angles.

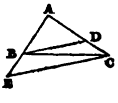


Produce any side B C, then the angle A C D is greater than either of the angles A or B (XVI), therefore A C B together with either A or B is less than the same angle A C B together with A C D; that is, less than two right angles (XIII). In the same manner, if C B be produced from the point B, it can be demonstrated that the angle A B C together with the angle A is less than two right angles; therefore any two angles of the triangle are less than two right angles.

This proposition and the sixteenth are included in the thirty-second, which proves that the three angles are together equal to two right angles.

Proposition XVIII. Theorem.

(96) In any triangle (B A C) if one side (A C) be greater than another (A B), the angle opposite to the greater side is greater than the angle opposite to the less.



From the greater side A C cut off the part A D equal to the less (III), and conterminous with it, and join B D.

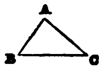
The triangle B A D being isosceles (V), the angles A B D and A D B are equal; but A D B is greater than the internal angle A C B (XVI): therefore A B D is greater than A C B, and therefore A B C is greater than A C B: but A B C is opposite the greater side A C, and A C B is opposite the less A B.

This proposition might also be proved by producing the lesser side A B, and taking A E equal to the greater side. In this case the angle A E C is equal to A C E (V), and therefore greater than A C B. But A B C is greater than A E C (XVI), and therefore A B C is greater than A C B.

Proposition XIX. Theorem.

(97) If in any triangle (B A C) one angle (B) be greater than another (C), the side (A C) which is opposite the greater angle is greater than the side (A B), which is opposite to the less.

For the side A C is either equal, or less, or greater than A B. It is not equal to A B, because the angle B would then be equal to C (V), which is contrary to the hypothesis.



It is not less than A B, because the angle B would then be less than C (XVIII), which is also contrary to the hypothesis.

Since therefore the side A C is neither equal to nor less than A B, it is greater than it.

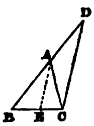
This proposition holds the same relation to the sixth, as the preceding does to the fifth. The four might be thus combined: one angle of a triangle is greater or less than another, or equal to it, according as the side opposed to the one is greater or less than, or equal to the side opposed to the other, and *vice versa.*

The student generally feels it difficult to remember which of the two, the eighteenth or nineteenth, is proved by construction, and which indirectly. By referring them to the fifth and sixth the difficulty will be removed.

Proposition XX.

(98) Any two sides (A B and A C) of a triangle (B A C) taken together, are greater than the third side (B C).

Let the side B A be produced, and let A D be cut off equal to A C (III), and let D C be drawn.



Since A D and A C are equal, the angles D and A C D are equal (V). Hence the angle B C D is greater than the angle D, and therefore the side B D in the triangle B C D is greater than B C (XIX). But B D is equal to B A and A C taken together, since A D was assumed equal to A C. Therefore B A and A C taken together are greater than B C.

This proposition is sometimes proved by bisecting the angle A. Let B E bisect it. The angle B E A is greater than E A C, and the angle C E A is greater than E A B (XVI); and since the parts of the angle A are equal, it follows, that each of the angles E is greater than each of the parts of A; and thence, by (XIX), it follows, that B A is greater than B E, and A C greater than C E, and therefore that the sum of the former is greater than the sum of the latter.

The proposition might likewise be proved by drawing a perpendicular from the angle A on the side B C; but these methods seem inferior in clearness and brevity to that of Euclid.

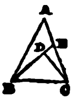
Some geometers, among whom may be reckoned Archimedes, ridicule this proposition as being self evident, and contend that it should be therefore one of the axioms. That a truth is considered self evident is, however, not a sufficient reason why it should be adopted as a geometrical axiom (55).

(99) It follows immediately from this proposition, that the difference of any two sides of a triangle is less than the remaining side. For the sides A C and B C together are greater than A B; let the side A C be taken from both, and we shall have the side B C greater than the remainder upon taking A C from A B; that is, then the difference between A B and A C.

In this proof we assume something more than is expressed in the fifth axiom. For we take for granted, that if one quantity (*a*) be greater than another (*b*), and that equals be taken from both, the remainder of the former (*a*) will be greater than the remainder of the latter (*b*). This is a principle which is frequently used, though not directly expressed in the axiom.

PROPOSITION XXI. THEOREM.

(100) The sum of two right lines (D B and D C) drawn to a point (D), within a triangle (B A C) from the extremities of any side (B C), is less than the sum of the two other sides of the triangle (A B and A C), but the lines contain a greater angle.



Produce B D to E. The sum of the sides B A and A E of the triangle B A E is greater than the third side B E (XX); add E C to each, and the sum of the sides B A and A C is greater than the sum of B E and E C, but the sum of the sides D E and E C of the triangle D E C is greater than the third side D C (XX); add B D to each, and the sum of B E and E C is greater than the sum of B D and D C, but the sum of B A and A C is greater than that of B E and E C; therefore the sum of B A and A C is greater than that of B D and D C.

Because the external angle B D C is greater than the internal D E C (XVI), and for the same reason D E C is greater than A, the angle B D C is greater than the angle A.

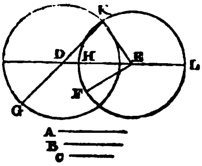
⁂ By the thirty-second proposition it will follow, that the angle B D C exceeds the angle A by the sum of the angles A B D and A C D. For the angle B D C is equal to the sum of D E C and D C E; and, again, the angle D E C is equal to the sum of the angles A and A B E. Therefore the angle B D C is equal to the sum of A, and the angles A B D and A C D.

PROPOSITION XXII. PROBLEM.

(101) Given three right lines (A, B, and C) the sum of any two of which is greater than the third, to construct a triangle whose sides shall be respectively equal to the given lines.

Solution.

From any point D draw the right line D E equal to one of the given lines A (II), and from the same point draw D G equal to another of the given lines B, and from the point E draw E F equal to C. From the centre D with the radius D G describe a circle, and from the centre E with the radius E F describe another circle, and from a point K of intersection of these circles draw K D and K E.



Demonstration.

It is evident, that the sides D E, D K, and K E of the triangle D K E are equal to the given right lines A, B, and C.

⁂ In this solution Euclid assumes that the two circles will have at least one point of intersection. To prove this, it is only necessary to show that a part of one of the circles will be within, and another part without the other (58).

Since D E and E K or E L are together greater than D K, it follows, that D L is greater than the radius of the circle K G, and therefore the point L is outside the circle. Also, since D K and E K are together greater than D E, if the equals E K and E H be taken from both, D H is less than D K, that is, D H is less than the radius of the circle, and therefore the point H is within it. Since the point H is within the circle and L without it, the one circle must intersect the other.

It is evident, that if the sum of the lines B and C were equal to the line A, the points H and K would coincide; for then the sum of D K and K E would equal D E. Also, if the sum of A and B were equal to C, the points K and L would coincide; for then E K would be equal to D K and D E, or to L D. It will hereafter appear, that in the former case the circles would touch externally, and in the latter internally.

If the line A were greater than the sum of B and C, it is easy to perceive that the circles would not meet, one being wholly outside the other; and if B were greater than the sum of A and C, they would not meet, one being wholly within the other.

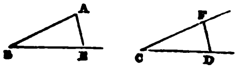
If the three right lines A B C be equal, this proposition becomes equivalent to the first, and the solution will be found to agree exactly with that of the first.

PROPOSITION XXIII. PROBLEM.

(102) At a given point (B) in a given right line (B E) to make an angle equal to a given angle (C).

Solution.

In the sides of the given angle take any points D and F; join D F, and construct a triangle E B A which shall be equilateral with the triangle D C F, and whose sides A B and E B meeting at the given point B shall be equal to F C and D C of the given angle C (XXII). The angle E B A is equal to the given angle D C F.



Demonstration.

For as the triangles D C F and E B A have all their sides respectively equal, the angles F C D and A B E opposite the equal sides D F and E A are equal (VIII).

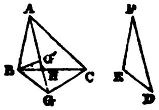
It is evident that the eleventh proposition is a particular case of this.

PROPOSITION XXIV. THEOREM.

(103) If two triangles (E F D, B A C) have two sides of the one respectively equal to two sides of the other (F E to A B and F D to A C), and if one of the angles (B A C) contained by the equal sides be greater than the other (E F D), the side (B C) which is opposite to the greater angle is greater than the side (E D) which is opposite to the less angle.

From the point A draw the right line A G, making with the side A B, which is not the greater, an angle B A G equal to the angle E F D (XXIII). Make A G equal to F D (III), and draw B G and G C.

In the triangles B A G and E F D the sides B A and A G are equal respectively to E F and F G, and the included angles are equal (const.), and therefore B G is equal to E D. Also, since A G is equal to F D by const., and A C is equal to it by hyp., A G is equal to A C, therefore the triangle G A C is isosceles, and therefore the angles A C G and A G C are equal (V); but the angle B G C is greater than A G C, therefore greater than A C G, and therefore greater than B C G; then in the triangle B G C the angle B G C is greater than B C G, therefore the side B C is greater than B G (XIX), but B G is equal to E D, and therefore B C is greater than E D.



In this demonstration it is assumed by Euclid, that the points A and G will be on different sides of B C, or, in other words, that A H is less than A G or A C. This may be proved thus:—The side A C not being less than A B, the angle A B C cannot be less than the angle A C B (XVIII). But the angle A B C must be less than the angle A H C (XVI); therefore the angle A C B is less than A H C, and therefore A H less than A C or A G (XIX).

In the construction for this proposition Euclid has omitted the words 'with the side which is not the greater.' Without these it would not follow that the point G would fall below the base B C, and it would be necessary to give demonstrations for the cases in which the point G falls on, or above the base B C. On the other hand, if these words be inserted, it is necessary in order to give validity to the demonstration, to prove as above, that the point G falls below the base.

If the words 'with the side not the greater' be not inserted, the two omitted cases may be proved as follows:

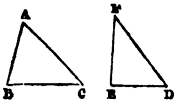
If the point G fall on the base B C, it is evident that B G is less than B C (51).

If G fall above the base B C let it be at G′. The sum of the lines B G′ and A G′ is less than the sum of A C and C B (XXI). The equals A C and A G′ being taken away, there will remain B G′ less than B C.

Proposition XXV. Theorem.

(104) **If two triangles (B A C and E F D) have two sides of the one respectively equal to two of the other (B A to E F and A C to F D), and if the third side of the one (B C) be greater than the third side (E D) of the other, the angle (A) opposite to the greater side is greater than the angle F, which is opposite to the less.**

The angle A is either equal to the angle F, or less than it, or greater than it.



It is not equal; for if it were, the side B C would be equal to the side E D (IV), which is contrary to the hypothesis.

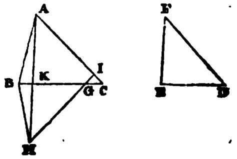
It is not less; for if it were, the side B C would be less than the side E D (XXIV), which is contrary to the hypothesis.

Since therefore the angle A is neither equal to, nor less than F, it must be greater.

This proposition might be proved directly thus:

On the greater side B C take B G equal to the lesser side E D, and on B G construct a triangle D H G equilateral with E F D. Join A H, and produce H G to I.

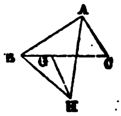
The angle H will then be equal to the angle F.



1° Let B G be greater than B K.

Since B A and B H are equal, the angles B A H and B H A are equal (V). Also since H G is equal to A C, it is greater than A I, and therefore H I is greater than A I, and therefore the angle H A I is greater than the angle A H I (XVIII). Hence, if the equal angles B H A and B A H be added to these, the angle B A C will be found greater than the angle B H G, which is equal to F.

2° If B G be not greater than B K, it is evident that the angle H is less than the angle A.



The twenty-fourth and twenty-fifth propositions are analogous to the fourth and eighth, in the same manner as the eighteenth and nineteenth are to the fifth and sixth. The four might be announced together thus:

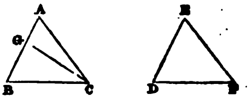
If two triangles have two sides of the one respectively equal to two sides of the other, the remaining side of the one will be greater or less than, or equal to the remaining side of the other, according as the angle opposed to it in the one is greater or less than, or equal to the angle opposed to it in the other, or *vice versa.*

In fact, these principles amount to this, that if two lines of given lengths be placed so that one pair of extremities coincide, and so that in their initial position the lesser line is placed upon the greater, the distance between the other extremities will then be the difference of the lines. If they be opened so as to form a gradually increasing angle, the line joining their extremities will gradually increase, until the angle they include becomes equal to two right angles, when they will be in one continued line, and the line joining their extremities is their sum. Thus the major and minor limits of this line is the *sum* and *difference* of the given lines. This evidently includes the twentieth proposition.

PROPOSITION XXVI. THEOREM.

(105) If two triangles (B A C, D E F) have two angles of the one respectively equal to two angles of the other (B to D and C to F), and a side of the one equal to a side of the other similarly placed with respect to the equal angles, the remaining sides and angles are respectively equal to one another.

First, let the equal sides be B C and D F, which lie between the equal angles; then the side B A is equal to the side D E.

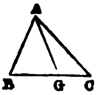


For if it be possible, let one of them B A be greater than the other; make B G equal to D E, and join C G.

In the triangles G B C, E D F the sides G B, B C are respectively equal to the sides E D, D F (const.), and the angle B is equal to the angle D (hyp.), therefore the angles B C G and D F E are equal (IV); but the angle B C A is also equal to D F E (hyp.), therefore the angle B C G is equal to B C A (51), which is absurd: neither of the sides B A and D E therefore is greater than the other, and therefore they are equal, and also B C and D F are equal (IV), and the angles B and D; therefore the side A C is equal to the side E F, as also the angle A to the angle E (IV).

Next, let the equal sides be B A and D E, which are opposite to the equal angles C and F, and the sides B C and D F shall also be equal.

For if it be possible, let one of them B C be greater than the other; make B G equal to D F, and join A G.



In the triangles A B G, E D F, the sides A B, B G are respectively equal to the sides E D, D F (const.), and the angle B is equal to the angle D (hyp.); therefore the angles A G B and E F D are equal (IV); but the angle C is also equal to E F D, therefore A G B and C are equal, which is absurd (XVI). Neither of the sides B C and D F is therefore greater than the other, and they are consequently equal. But B A and D E are also equal, as also the angles B and D; therefore the side A C is equal to the side E F, and also the angle A to the angle E (IV).

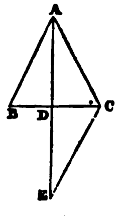
It is evident that the triangles themselves are equal in every respect.

⁂ (106) Cor. 1.—From this proposition and the principles previously established, it easily follows, that a line being drawn from the vertex of a triangle to the base, if any two of the following equalities be given (except the first two), the others may be inferred.

1° The equality of the sides of the triangle.
2° The equality of the angles at the base.
3° The equality of the angles under the line drawn, and the base.
4° The equality of the angles under the line drawn, and the sides.
5° The equality of the segments of the base.

Some of the cases of this investigation have already been proved. (74), (75), (76). The others present no difficulty, except in the case where the fourth and fifth qualities are given to infer the others. This case may be proved as follows.

If the line A D, which bisects the vertical angle (A) of a triangle also bisect the base B C, the triangle will be isosceles; for produce A D so that D E shall be equal to A D, and join E C. In the triangles D C E and A D B the angles vertically opposed at D are equal, and also the sides which contain them; therefore (IV) the angles B A D and D E C are equal, and also the sides A B and E C. But the angle B A D is equal to D A C (hyp.); and therefore D A C is equal to the angle E, therefore (VI) the sides A C and E C are equal. But A B and E C have been already proved equal, and therefore A B and A C are equal.



⁂ (107) The twenty-sixth proposition furnishes the third criterion which has been established in the Elements for the equality of two triangles. It may be observed, that in a triangle there are six quantities which may enter into consideration, and in which two triangles may agree or differ; viz. the three sides and the three angles. We can in most cases infer the equality of two angles in every respect, if they agree in any three of those six quantities *which are independent of each other.* To this, however, there are certain exceptions, as will appear by the following general investigation of the question.

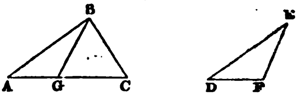
When two triangles agree in three of the six quantities already mentioned, these three must be some of the six following combinations:

1° Two sides and the angle between them.
2° Two angles and the side between them.
3° Two sides, and the angle opposed to one of them.
4° Two angles, and the side opposed to one of them.
5° The three sides.
6° The three angles.

The first case has been established in the fourth, and the second and fourth in the twenty-sixth proposition The fifth case has been established by the eighth, and in the sixth case the triangles are not necessarily equal. In this case, however, the three data are not independent, for it will appear by the thirty-second proposition, that any one angle of a triangle can be inferred from the other two.

The third is therefore the only case which remains to be investigated.

⁂ (108) 3° *To determine under what circumstance two triangles having two sides equal, each to each, and the angles opposed to one pair of equal sides equal, shall be equal in all respects.* Let the sides A B and B C be equal to D E and E F, and the angle A be equal to the angle D. If the angles B and E be equal, it is evident that the triangles are in every respect equal, by (IV), and that C and F are equal. But if B and E be not equal, let one (B) be greater than the other E; and from B let a line B G be drawn, making the angle A B G equal to the angle E. In the triangles A B G and D E F, the angles A and A B G are equal respectively to D and E, and the side A B is equal to D E, therefore (XXVI) the triangles are in every respect equal; and the side B G is equal to E F, and the angle B G A equal to the angle F. But since



E F is equal to B C, B G is equal to B C, and therefore (V) B G C is equal to B C G, and therefore C and G B A or F are supplemental.

(109) Hence, *if two triangles have two sides in the one respectively equal to two sides in the other, and the angles opposed to one pair of equal sides equal, the angles opposed to the other equal sides will be either equal or supplemental.*

⁂ (110) Hence it follows, that if two triangles have two sides respectively equal each to each, and the angles opposed to one pair of equal sides equal, the remaining angles will be equal, and therefore the triangles will be in every respect equal, if there be any circumstance from which it may be inferred that the angles opposed to the other pair of equal sides are of the same species.

(Angles are said to be of the same species when they are both acute, both obtuse, or both right).

For in this case if they be not right they cannot be supplemental, and must therefore be equal (109), in which case the triangles will be in every respect equal, by (XXVI).

If they be both right, the triangles will be equal by (108); because in that case G and C being right angles, B G must coincide with B C, and the triangle B G A with B C A; but the triangle B G A is equal to E F D, therefore, &c.

⁂ (111) There are several circumstances which may determine the angles opposed to the other pair of equal sides to be of the same species, and therefore which will determine the equality of the triangles; amongst which are the following:

If one of the two angles opposed to the other pair of equal sides be right; for a right angle is its own supplement.

If the angles which are given equal be obtuse or right; for then the other angles must be all acute (91), and therefore of the same species.

If the angles which are included by the equal sides be both right or obtuse; for then the remaining angles must be both acute.

If the equal sides opposed to angles which are not given equal be less than the other sides, these angles must be both acute (XVIII).

In all these cases it may be inferred, that the triangles are in every respect equal.

It will appear by prop. 38, that if two triangles have two sides respectively equal, and the included angles supplemental, their areas are equal.

(The *area* of a figure is the quantity of surface within its perimeter).

(112) If several right lines be drawn from a point to a given right line.

1° The shortest is that which is perpendicular to it.

2° Those equally inclined to the perpendicular are equal, and *vice versa.*

3° Those which meet the right line at equal distances from the perpendicular are equal, and *vice versa.*

4° Those which make greater angles with the perpendicular are greater, and *vice versa.*

5° Those which meet the line at greater distances from the perpendicular are greater, and *vice versa.*

6° More than two equal right lines cannot be drawn from the same point to the same right line.

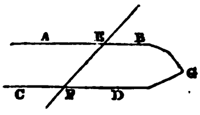
The student will find no difficulty in establishing these principles.

(113) If any number of isosceles triangles be constructed upon the same base, their vertices will be all placed upon the right line, which is perpendicular to the base, and passes through its middle point. This is a very obvious and simple example of a species of theorem which frequently occurs in geometrical investigations. This perpendicular is said to be the *locus* of the vertex of isosceles triangles standing on the same base.

Proposition XXVII. Theorem.

(114) If a line (E F) intersect two right lines (A B and C D), and make the alternate angles equal to each other (A E F to E F D), these right lines are parallel.

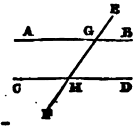
For, if it be possible, let those lines not be parallel but meet in G; the external angle A E F of the triangle E G F is greater than the internal E F G (XVI); but it is also equal to it (by hyp.), which is absurd; therefore A B and C D do not meet at the side B D; and in the same manner it can be demonstrated, that they do not meet at the side A C; since, then, the right lines do not meet on either side they are parallel.



Proposition XXVIII. Theorem.

(115) If a line (E F) intersect two right lines (A B and C D), and make the external angle equal to the internal and opposite angle on the same side of the line (E G A to G H C, and E G B to G H D); or make the internal angles at the same side (A G H and C H G or B G H and D H G) equal together to two right angles, the two right lines are parallel to another.

First, let the angles E G A and G H C be equal; and since the angle E G A is equal to B G H (XV), the angles G H C and B G H are equal; but they are the alternate angles, therefore the right lines A B and C D are parallel (XXVII).



In the same manner the proposition can be demonstrated, if the angles E G B and G H D were given equal.

Next, let the angles A G H and C H G taken together be equal to two right angles; since the angles G H D and G H C taken together are also equal to two right angles (XIII), the angles A G H and C H G taken together are equal to the angles G H D and C H G taken together; take away the common angle C H G and the remaining angle A G H is equal to G H D; but they are the alternate angles, and therefore the right lines A B and C D are parallel (XXVII). In the same manner the proposition can be demonstrated, if the angles B G H and D H G were given equal to two right angles.

By this proposition it appears, that if the line G B makes the angle B G H equal to the supplement of G H D (84), the line G B will be parallel to H D. In the twelfth axiom (54) it is *assumed*, that if a line make an angle with G H less than the supplement of G H D, that line will not be parallel to H D, and will therefore meet it, if produced. The principle, therefore, which is really assumed is, that two right lines which intersect each other cannot be both parallel to the same right line, a principle which seems to be nearly self-evident.

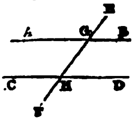
If it be granted that the two right lines which make with the third, G H, angles less than two right angles be not parallel, it is plain that they must meet on that side of G H on which the angles are less than two right angles; for the line passing through G, which makes a less angle than B G H, with G H on the side B D, will make a greater angle than A G H with G H on the side A C; and therefore that part of the line which lies on the side A C will lie above A G, and therefore can never meet H C.

Various attempts have been made to supersede the necessity of assuming the twelfth axiom; but all that we have ever seen are attended with still greater objections. Neither does it seem to us, that the principle which is really assumed as explained above can reasonably be objected against.

PROPOSITION XXIX. THEOREM.

(116) If a right line (E F) intersect two parallel right lines (A B and C D), it makes the alternate angles equal (A G H to G H D, and C H G to H G B); and the external angle equal to the internal and opposite upon the same side (E G A to G H C, and E G B to G H D); and also the two internal angles at the same side (A G H and C H G, B G H and D H G) together equal to two right angles.

1° The alternate angles A G H and G H D are equal; for if it be possible, let one of them A G H be greater than the other, and adding the angle B G H to both, A G H and B G H together are greater than B G H and G H D; but A G H and B G H together are equal



to two right angles (XIII), therefore B G H and G H D are less than two right angles, and therefore the lines A B and C D, if produced, would meet at the side B D (Axiom 12); but they are parallel (hyp.), and therefore cannot meet, which is absurd. Therefore neither of the angles A G H and G H D is greater than the other; they are therefore equal.

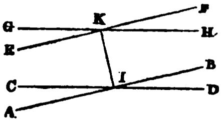
In the same manner it can be demonstrated, that the angles B G H and G H C are equal.

2° The external angle E G B is equal to the internal G H D; for the angle E G B is equal to the angle A G H (XV); and A G H is equal to the alternate angle G H D (first part); therefore E G B is equal to G H D. In the same manner it can be demonstrated, that E G A and G H C are equal.

3° The internal angles at the same side B G H and G H D together are equal to two right angles; for since the alternate angles G H D and A G H are equal (first part), if the angle B G H be added to both, B G H and G H D together are equal to B G H and A G H, and therefore are equal to two right angles (XIII). In the same manner it can be demonstrated, that the angles A G H and G H C together are equal to two right angles.

(117) Cor. 1.—If two right lines which intersect each other (A B, C D) be parallel respectively to two others (E F, G H), the angles included by those lines will be equal.

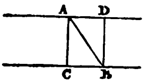
Let the line I K be drawn joining the points of intersection. The angles C I K and I K H are equal, being alternate; and the angles A I K and I K F are equal, for the same reason. Taking the former from the latter the angles A I C and H K F remain equal. It is evident that their supplements C I B and G K F are also equal.



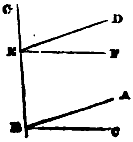
(118) Cor. 2.—If a line be perpendicular to one of two parallel lines, it will be also perpendicular to the other; for the alternate angles must be equal.

(119) Cor. 3.—The parts of all perpendiculars to two parallel lines intercepted between them are equal.

For let A B be drawn. The angles B A C and A B D are equal, being alternate; and the angles B A D and A B C are equal, for the same reason; the side A B being common to the two triangles, the sides A C and B D must be equal (XXVI).



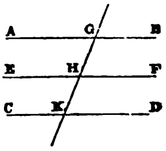
(120) Cor. 4.—If two angles be equal (A B C and D E F), and the sides A B and D E be parallel, and the other sides B C and E F lie at the same side of them, they will also be parallel; for draw B E. Since A B and D E are parallel, the angles G B A and G E D are equal. But, by hypothesis, the angles A B C and D E F are equal; adding these to the former, the angles G B C and G E F are equal. Hence the lines B C and E F are parallel.



PROPOSITION XXX. THEOREM.

(121) If two right lines (A B, C D) be parallel to the same right line (E F), they are parallel to each other.

Let the right line G K intersect them; the angle A G H is equal to the angle G H F (XXIX); and also the angle H K D is equal to G H F (XXIX); therefore A G H is equal to G K D; and therefore the right lines A B and C D are parallel.



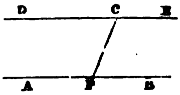
(122) Cor.—Hence two parallels to the same line cannot pass through the same point. This is, in fact, equivalent to the twelfth axiom (115).

PROPOSITION XXXI. PROBLEM.

(123) Through a given point (C) to draw a right line parallel to a given right line (A B).

Solution.

In the line A B take any point F, join C F, and at the point C and with the right line C F make the angle F C E equal to A F C (XXIII), but at the opposite side of the line C F; the line D E is parallel to A B.



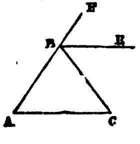
Demonstration.

For the right line F C intersecting the lines D E and A B makes the alternate angles E C F and A F C equal, and therefore the lines are parallel (XXVII).

PROPOSITION XXXII. THEOREM.

(124) If any side (A B) of a triangle (A B C) be produced, the external angle (F B C) is equal to the sum of the two internal and opposite angles (A and C); and the three internal angles of every triangle taken together are equal to two right angles.

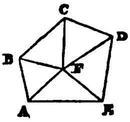
Through B draw B E parallel to A C (XXXI.) The angle F B E is equal to the internal angle A (XXIX), and the angle E B C is equal to the alternate C (XXIX); therefore the whole external angle F B C is equal to the two internal angles A and C.



The angle A B C with F B C is equal to two right angles (XIII); but F B C is equal to the two angles A and C (first part); therefore the angle A B C together with the angles A and C is equal to two right angles.

(125) Cor. 1.—If one angle of a triangle be right, the sum of the other two is equal to a right angle.
(126) Cor. 2.—If one angle of a triangle be equal to the sum of the other two angles, that angle is a right angle.
(127) Cor. 3.—An obtuse angle of a triangle is greater and an acute angle less than the sum of the other two angles.
(128) Cor. 4.—If one angle of a triangle be greater than the sum of the other two it must be obtuse; and if it be less than the sum of the other two it must be acute.
(129) Cor. 5.—If two triangles have two angles in the one respectively equal to two angles in the other, the remaining angles must be also equal.
(130) Cor. 6.—Isosceles triangles having equal vertical angles must also have equal base angles.
(131) Cor. 7.—Each base angle of an isosceles triangle is equal to half the external vertical angle.
(132) Cor. 8.—The line which bisects the external vertical angle of an isosceles triangle is parallel to the base, and *vice versa.*
(133) Cor. 9.—In a right-angled isosceles triangle each base angle is equal to half a right angle.

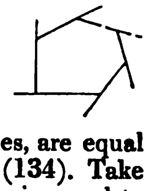
(134) Cor. 10.—All the internal angles of any rectilinear figure A B C D E, together with four right angles, are equal to twice as many right angles as the figure has sides.



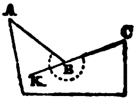
Take any point F within the figure, and draw the right lines F A, F B, F C, F D, and F E. There are formed as many triangles as the figure has sides, and therefore all their angles taken together are equal to twice as many right angles as the figure has sides (XXXII); but the angles at the point F are equal to four right angles (83); and therefore the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

This is the first corollary in the Elements, and the following is the second.

(135) Cor. 11.—The external angles of any rectilinear figure are together equal to four right angles: for each external angle, with the internal adjacent to it, is equal to two right angles (XIII); therefore all the external angles with all the internal are equal to twice as many right angles as the figure has sides; but the internal angles, together with four right angles, are equal to twice as many right angles as the figure has sides (134). Take from both, the internal angles and the external remain equal to four right angles.



⁂ This corollary is only true of what are called *convex figures;* that is, of figures in which every internal angle is less than two right angles. Some figures, however, have angles which are called *reentrant* angles, and which are greater than two right angles. Thus in this figure the angle A B C exceeds two right angles, by the angle K B A, formed by the side B A with the production of the side B C. This angle K B A is that which in ordinary cases is the external angle, but which in the present instance constitutes a part of the internal angle, and in this case there is no external angle. The angle which is considered as the reentrant angle, and one of the internal angles of the figure is marked with the dotted curve in the figure. See (14).



⁂ (136) A figure which has no reentrant angle is called a *convex figure.*

It should be observed, that the first corollary applies to all rectilinear figures, whether convex or not, but the second only to convex figures.

⁂ (137) If a figure be not convex each reentrant angle exceeds two right angles by a certain excess, and has no adjacent external angle, while each ordinary angle, together with its adjacent external angle, is equal to two right angles. Hence it follows, that the sum of all the angles internal and external, including the reentrant angles, is equal to twice as many right angles as the figure has sides, together with the excess of every reentrant angle above two right angles. But (134) the sum of the internal angles alone is equal to twice as many right angles as the figure has sides, deducting four; hence the sum of the external angles must be equal to those four right angles, together with the excess of every reentrant angle above two right angles.

The sum of the external angles of every convex figure must be the same; and, however numerous the sides and angles be, this sum can never exceed four right angles.

If every pair of alternate sides of a convex figure be produced to meet, the sum of the angles so formed will be equal to $2n - 8$ right angles. This may be proved by showing that each of these angles with two of the external angles is equal to two right angles.

⁂ (138) Cor. 12.—The sum of the internal angles of a figure is equal to a number of right angles expressed by twice the number of sides, deducting four; also as each reentrant angle must be greater than two right angles, the sum of the reentrant angles must be greater than twice as many right angles as there are reentrant angles. Hence it follows, that twice the number of sides deducting four, must be greater than twice the number of reentrant angles, and therefore that the number of sides deducting two, must be greater than the number of reentrant angles; from which it appears, that the number of reentrant angles in a figure must always be at least three less than the number of sides. There must be therefore at least three angles in every figure, which are each less than two right angles.

⁂ (139) Cor. 13.—A triangle cannot therefore have any reentrant angle, which also follows immediately from considering that the three angles are together equal to two right angles, while a single reentrant angle would be greater than two right angles.

⁂ (140) Cor. 14.—No equiangular figure can have a reentrant angle, for if one angle were reentrant all should be so, which cannot be (138).

*** (141) Cor. 15.—If the number of sides in an equiangular figure be given, the magnitude of its angles can be determined. Since it can have no reentrant angle, the sum of its external angles is equal to four right angles; the magnitude of each external angle is therefore determined by dividing four right angles by the number of sides. This being deducted from two right angles, the remainder will be the magnitude of each angle. Thus the fraction whose numerator is 4, and whose denominator is the number of sides, expresses the part of a right angle which is equal to the external angle of the figure, and if this fraction be deducted from the number 2, the remainder will express the internal angle in parts of a right angle. In the notation of arithmetic, if n be the number of sides, the external angle is the $\frac{4}{n}$th and the internal angle the $(2 - \frac{4}{n})^{\text{th}}$ of a right angle.

*** (142) Cor. 16.—The sum of the angles of every figure is equal to an even number of right angles. For twice the number of sides is necessarily even, and the even number four being subducted leaves an even remainder. Hence it appears, that no figure can be constructed the sum of whose angles is equal to 3, 5, or 7 right angles, &c.

*** (143) Cor. 17.—If the number of right angles to which the sum of the angles of any figure is equal be given, the number of sides may be found. For since the number of right angles increased by four is equal to twice the number of sides, it follows, that half the number of right angles increased by two is equal to the number of sides.

*** (144) Cor. 18.—If all the angles of a figure be right, it must be a quadrilateral, and therefore a right angled parallelogram. For (141) the magnitude of each external angle is determined in parts of a right angle by dividing 4 by the number of sides; in the present case each external angle must be a right angle, and therefore 4 divided by the number of sides must be 1, and therefore the number of sides must be four. Each of the four angles being right, every adjacent pair is equal to two right angles, and therefore the opposite sides of the figure are parallel.

*** (145) Cor. 19.—The angle of an equilateral triangle is equal to one third of two right angles, or two thirds of a right angle.

That one third of two right angles is equal to two thirds of one right angle, easily appears from considering that as three thirds of a right angle is equal to one right angle, six thirds will be equal to two right angles, and one third of this is two thirds of one right angle.

(146) Cor. 20.—To trisect a right angle. Construct any equilateral triangle and draw a line (XXIII), cutting off from the given angle an angle equal to an angle of the equilateral triangle. This angle being two thirds of the whole, if it be bisected, the whole right angle will be trisected.

By the combination of bisection and trisection a right angle may be divided into 2, 3, 4, 6, 8, &c. equal parts.

N. B. The general problem to trisect *any angle* is one which has never been solved by plane Geometry.

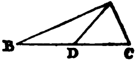
*** (147) Cor. 21.—The multisection of a right angle may be extended by means of the angles of the regular polygons.

In a regular pentagon the external angle is four fifths of a right angle; the complement of this angle being the fifth of a right angle solves the problem to divide a right angle into five equal parts.

In a regular heptagon the external angle is four sevenths of a right angle, which being divided into four equal parts (IX) gives the seventh of a right angle, and solves the problem to divide a right angle into seven equal parts.

Thus in general the problem of the multisection of a right angle is resolved to that of the construction of the regular polygons, and *vice versa*. On this subject the student is referred to the fourth book of the Elements.

(148) Cor. 22.—The vertical angle A of a triangle is right, acute or obtuse, according as the line A D which bisects the base B C is equal to, greater or less than half the base B D.

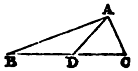


1. If the line A D be equal to half the base B D, the triangles A D B and A D C will be isosceles, therefore the angles B A D and C A D will be respectively equal to the angles B and C. The angle A is therefore equal to the sum of B and C, and is therefore (126) a right angle.

2. If A D be greater than B D or D C, the angles B A D and C A D are respectively less than the angles B and C, and therefore the angle A is less than the sum of B and C, and is therefore (128) acute.

3 If A D be less than B D or D C, the angles B A D and C A D are respectively greater than B and C, and therefore the angle A is greater than the sum of B and C, and is therefore (128) obtuse.

(149) Cor. 23.—The line drawn from the vertex A of a triangle bisecting the base B C is equal to, greater or less than half the base, according as the angle A is right, acute, or obtuse.



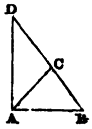
1. Let the angle A be right. Draw A D so that the angle B A D shall be equal to the angle B. The line B D will then bisect B C, and be equal to half of it.

For the angles B and C are together equal to the angle A (125), and since B is equal to B A D, C must be equal to C A D. Hence it follows, (VI) that B D A and C D A are isosceles triangles, and that B D and C D are equal to A D and to each other.

2. Let A be acute, and draw A D bisecting B C. The line A D must be greater than B D or D C; for if it were equal to them the angle A would be right, and if it were less it would be obtuse (148).

3. Let A be obtuse, and draw A D bisecting B C. The line A D must be less than each of the parts B D, D C; for if it were equal to them the angle A would be right, and if it were greater the angle A would be acute (148).

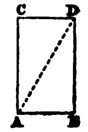
(150) Cor. 24.—To draw a perpendicular to a given right line through its extremity without producing it.



Take a part A B from the extremity A, and construct on it an equilateral triangle A C B. Produce B C so that C D shall be equal to A C, and draw D A. This will be the perpendicular required. For since A C bisects B D, and is equal to half of it, the angle D A B is right (148).

PROPOSITION XXXIII. THEOREM.

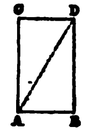
(150) Right lines (A C and B D) which join the adjacent extremities of two equal and parallel right lines (A B and C D) are themselves equal and parallel.



Draw the diagonal A D, and in the triangles C D A and B A D the sides C D and B A are equal (by hyp.); A D is common to both triangles, and the angle C D A is equal to the alternate B A D (XXIX); therefore the lines A C and B D are equal, and also the angles C A D and B D A; therefore the right line A D cutting the right lines A C and B D makes the alternate angles equal, and therefore (XXVII) the right lines A C and B D are parallel.

PROPOSITION XXXIV. THEOREM.

(151) The opposite sides (A B and C D, A C and B D) of a parallelogram (A D) are equal to one another, as are also the opposite angles (A and D, C and B), and the parallelogram itself is bisected by its diagonal (A D).



For in the triangles C D A, B A D, the alternate angles C D A and B A D, C A D and B D A are equal to one another (XXIX), and the side A D between the equal angles is common to both triangles; therefore the sides C D and C A are equal to A B and B D (XXVI), and the triangle C D A is equal to the triangle B A D, and the angles A C D and A B D are also equal; and since the angle A C D with C A B is equal to two right angles (XXIX), and A B D with C D B is equal to two right angles, take the equals A C D and A B D from both, and the remainders C A B and C D B are equal.

(152) COR. 1.—If two parallelograms have an angle in the one equal to an angle in the other, all the angles must be equal each to each. For the opposite angles are equal by this proposition, and the adjacent angles are equal, being their supplements.

(153) COR. 2.—If one angle of a parallelogram be right, all its angles are right; for the opposite angle is right by (151), and the adjacent angles are right, being the supplements of a right angle.

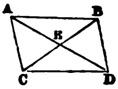
(154) Both diagonals A D, B C being drawn, it may, with a few exceptions, be proved that a quadrilateral figure which has any *two* of the following properties will also have the others:

1° The parallelism of A B and C D.
2° The parallelism of A C and B D.
3° The equality of A B and C D.
4° The equality of A C and B D.
5° The equality of the angles A and D.
6° The equality of the angles B and C.
7° The bisection of A D by B C.
8° The bisection of B C by A D.
9° The bisection of the area by A D.
10° The bisection of the area by B C.

These ten data combined in pairs will give 45 distinct pairs; with each of these pairs it may be required to establish any of the eight other properties, and thus 360 questions respecting such quadrilaterals may be raised. These questions will furnish the student with an useful geometrical exercise. Some of the most remarkable cases are among the following corollaries:

The 9th and 10th data require the aid of subsequent propositions.

(155) Cor. 3.—*The diagonals of a parallelogram bisect each other.*

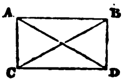


For since the sides A C and B D are equal, and also the angles C A E and B D E, as well as A C E and D B E, the sides (XXVI) C E and B E, and also A E and E D are equal.

(156) Cor. 4.—*If the diagonals of a quadrilateral bisect each other, it will be a parallelogram.*

For since A E and E C are respectively equal to D E and E B, and the angles A E C and D E B (XV) are also equal, the angles A C E and D B E are equal (IV); and, therefore, the lines A C and B D are parallel, and, in like manner, it may be proved that A B and C D are parallel.

(157) Cor. 5.—*In a right angled parallelogram the diagonals are equal.*



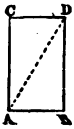
For the adjacent angles A and B are equal, and the opposite sides A C and B D are equal, and the side A B is common to the two triangles C A B and A B D, and therefore (IV) the diagonals A D and C B are equal.

If the diagonals of a parallelogram be equal, it will be right angled.

For in that case the three sides of the triangle C A B are respectively equal to those of D B A, and therefore (VIII) the angles A and B are equal. But they are supplemental, and therefore each is a right angle.

⁂ (158) The converses of the different parts of the 34th proposition true, and may be established thus:

If the opposite sides of a quadrilateral be equal it is a parallelogram.



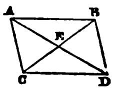
For draw A D. The sides of the triangles A C D and A B D are respectively equal, and therefore (VIII) the angles C A D and A D B are equal, and also the angles C D A and D A B. Hence the sides A C and B D, and also the sides A B and C D are parallel.

Hence a lozenge is a parallelogram, and a square has all its angles right.

If the opposite angles of a quadrilateral be equal, it will be a parallelogram.

For all the angles together are equal to four right angles (134); and since the opposite angles are equal, the adjacent angles are equal to half the sum of all the angles, that is, to two right angles, and therefore (XXVIII) the opposite sides are parallel.

If each of the diagonals bisect the quadrilateral, it will be a parallelogram.



This principle requires the aid of the 39th proposition to establish it. The triangles C A D and C B D are equal, each being half the whole area, therefore (XXXIX) the lines A B and C D are parallel. In the same manner D A B and D C B are equal, and therefore A C and B D are parallel.

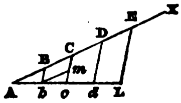
⁂ (159) *The diagonals of a lozenge bisect its angles.*

For each diagonal divides the lozenge into two isosceles triangles whose sides and angles are respectively equal.

⁂ (160) *If the diagonals of a quadrilateral bisect its angles, it will be a lozenge.*

For each diagonal in that case divides the figure into two triangles, having a common base placed between equal angles, and therefore (VI) the conterminous sides of the figure are equal.

⁂ (161) *To divide a finite right line A L into any given number of equal parts.*



From the extremity A draw any right line A X of indefinite length, and take upon it any part A B. Assume B C, C D, D E, &c. successively equal to A B (III), and continue this until a number of parts be assumed on A X equal in number to the parts into which it is required to divide A L. Join the extremity of the last part E with the extremity L, and through B C D, &c. draw parallels to E L. These parallels will divide A L into the required number of equal parts.

It is evident that the number of parts is the required number.

But these parts are also equal. For through *b* draw *b m* parallel to A E, and *b* c is a parallelogram; therefore *b m* is equal to B C or to A B. Also the angle A is equal to the angle *c b m,* and A *b* B to *b c m.* Hence (XXVI) A *b* and *b c* are equal. In like manner it may be proved, that *b c* and *c d* are equal, and so on.

(162) Parallelograms whose sides and angles are equal are themselves equal. For the triangles into which they are divided by their diagonals have two sides and the included angles respectively equal, and are therefore (IV) equal, and therefore their doubles, the parallelograms, are equal.

(163) Hence the squares of equal lines are equal.

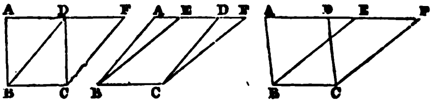
(164) Also equal squares have equal sides. For the diagonals being drawn, the right angled isosceles triangles into which they divide the squares are equal; the sides of these triangles must be equal, for if not let parts be cut off from the greater equal to the less, and their extre-

mities being joined, an isosceles right angled triangle will be found equal to the isosceles right angled triangle whose base is the diagonal of the other square (IV), and therefore equal to half of the other square, and also equal to half of the square a part of which it is; thus a part of the half square is equal to the half square itself, which is absurd.

PROPOSITION XXXV.

(165) Parallelograms on the same base (B C) and between the same parallels are equal.

For the angles B A F and C D F and also B E A and C F D are equal (XXIX), and the sides A B and D C are also equal (XXXIV), and therefore (XXVI) the triangles B A E and C D F are equal. These being successively taken from the whole quadrilateral B A F C, leave the remainders, which are the parallelograms B D and B F, equal.



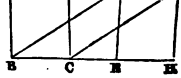
We have in this proof departed from Euclid in order to avoid the subdivision of the proposition into cases. The equality which is expressed in this and the succeeding propositions is merely equality of *area*, and not of sides or angles. The mere equality of area is expressed by *Legendre* by the word *equivalent*, while the term *equal* is reserved for equality in all respects. We have not thought this of sufficient importance however to justify any alteration in the text.

PROPOSITION XXXVI. THEOREM.

(166) Parallelograms (B D and E G) on equal bases and between the same parallels are equal.

Draw the right lines B F and C G.

Because the lines B C and F G are equal to the same E H (XXXIV), they are equal to one another; but they are also parallel, therefore B F and C G which join their extremities are parallel (XXXIII); and B G is a parallelogram, therefore equal to both B D and E G (XXXV), and therefore the parallelograms B D and E G are equal.

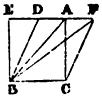


It is here supposed that the equal bases are placed in the same right line.

(167) COR.—If two opposite sides of a parallelogram be divided into the same number of equal parts, and the corresponding points of division be joined by right lines, these right lines will severally divide the parallelogram into as many equal parallelograms.

PROPOSITION XXXVII. THEOREM.

(168) Triangles (B A C and B F C) on the same base and between the same parallels are equal.



Through the point B draw B E parallel to C A, and draw B D parallel to C F, and produce A E to meet these lines at E and D. The figures B E A C and B D F C are parallelograms on the same base B C and between the same parallels, and therefore, (XXXV) equal; and the triangles B A C and B F C are their halves (XXXIV), and therefore also equal.

PROPOSITION XXXVIII.

(169) Triangles on equal bases and between the same parallels are equal.

For by the same construction as in the last proposition they are shown to be the halves of parallelograms on equal bases and between the same parallels.

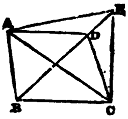
⁂ (170) COR. 1.—Hence a right line drawn from the vertex of a triangle bisecting the base bisects the area.

This proves that if two triangles have two sides respectively equal, and the included angles supplemental, the areas will be equal; for the two triangles into which the bisector of the base divides the triangle are thus related.

⁂ (171) COR. 2.—In general, if the base of a triangle be divided into any number of equal parts (161) lines drawn from the vertex to the several points of division will divide the area of the triangle into as many equal parts.

PROPOSITION XXXIX. THEOREM.

(172) Equal triangles (B A C and B D C) on the same base and on the same side of it are between the same parallels.

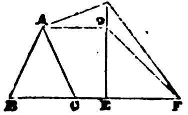


For if the right line A D which joins the vertices of the triangles be not parallel to B C, draw through the point A a right line A E parallel to B C, cutting a side B D of the triangle B D C or the side produced in a point E different from the vertex, and draw C E.

Because the right lines A E and B C are parallel, the triangle B E C is equal to B A C (XXXVII); but B D C is also equal to B A C (hyp.), therefore B E C and B D C are equal; a part equal to the whole, which is absurd. Therefore the line A E is no parallel to B C; and in the same manner it can be demonstrated, that no other line except A D is parallel to it, therefore A D is parallel to B C.

PROPOSITION XL. THEOREM.

(173) Equal triangles (B A C and E D F) on equal bases and on the same side, are between the same parallels.



For if the right line A D which joins the vertices of the two triangles be not parallel to B F, draw through the point A the right line A G parallel to B F, cutting a side D E of the triangle E D F, or the side produced in a point G different from the vertex, and join F G.

Because the right line A G is parallel to B F, and B C and E F are equal, the triangle G E F is equal to B A C (XXXVIII); but E D F is also equal to B A C (hyp.), therefore E G F and E D F are equal; a part equal to the whole, which is absurd. Therefore A G is not parallel to B F, and in the same manner it can be demonstrated, that no other line except A D is parallel to B F, therefore A D is parallel to B F.

From this and the preceding propositions may be deduced the following corollaries.

(174) Cor. 1.—Perpendiculars being drawn through the extremities of the base of a given parallelogram or triangle, and produced to meet the opposite side of the parallelogram or a parallel to the base of the triangle through its vertex, will include a right angled parallelogram which shall be equal to the given parallelogram; and if the diagonal of this right angled parallelogram be drawn, it will cut off a right angled triangle having the same base with the given triangle and equal to it. Hence any parallelogram or triangle is equal to a right angled parallelogram or triangle having an equal base and altitude.

(175) Cor. 2.—Parallelograms and triangles whose bases and altitudes are respectively equal are equal in area.

(176) Cor. 3.—Equal parallelograms and triangles on equal bases have equal altitudes.

(177) Cor. 4.—Equal parallelograms and triangles in equal altitudes have equal bases.

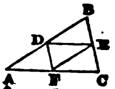
(178) Cor. 5.—If two parallelograms or triangles have equal altitudes, and the base of one be double the base of the other, the area of the one will be also double the area of the other. Also if they have equal bases and the altitude of one be double the altitude of the other, the area of the one will be double the area of the other.

(179) Cor. 6.—The line joining the points of bisection of the sides of a triangle is parallel to its base.

For if lines be drawn from the extremities of the base to the points of bisection they will each bisect the area (171) of the triangle; therefore the triangles having the base of the given triangle as a common base, and their vertices at the middle points of the sides, are equal, and therefore between the same parallels.

(180) Cor. 7.—*A parallel to the base of a triangle through the point of bisection of one side will bisect the other side.*

For by the last Cor. the line joining the points of bisection of the sides is parallel to the base, and two parallels to the same line cannot pass through the same point.



(181) Cor. 8.—*The lines which join the middle points* D E F *of the three sides of a triangle divide it into four triangles which are equal in every respect.*

(182) Cor. 9.—*The line joining the points of bisection of each pair of sides is equal to half of the third side.*

⁂ (183) Cor. 10.—If two conterminous sides of a parallelogram be divided each into any number of equal parts, and through the several points of division of each side parallels be drawn to the other side, the whole parallelogram will be divided into a number of equal parallelograms, and this number is found by multiplying the number of parts in one side by the number of parts in the other. This is evident from considering, that by the parallels through the points of division of one side the whole parallelogram is resolved into as many equal parallelograms as there are parts in the side through the points of which the parallels are drawn; and the parallels through the points of division of the other side resolve each of these component parallelograms into as many equal parallelograms as there are parts in the other side. Thus the total number of parallelograms into which the entire is divided, is the product of the number of parts in each side.

⁂ (184) Cor. 11.—The square of a line is four times the square of its half.

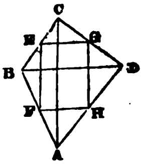
⁂ (185) Cor. 12.—If the sides of a right angled parallelogram be divided into any number of equal parts, and such that the parts of one side shall have the same magnitude as those of the other, the whole parallelogram will be equal to the square of one of the parts into which the sides are divided, multiplied by the product of the number of parts in each side. Thus, if the base of the parallelogram be six feet and the altitude be eight feet, the area will be one square foot multiplied by the product of six and eight or forty-eight square feet. In this sense the area of such a parallelogram is said to be found by multiplying its base by its altitude.

⁂ (186) Cor. 13.—Also, since the area of any parallelogram is equal to that of a right angled parallelogram having the same base and altitude, and that of a triangle is equal to half that area, it follows that the area of a parallelogram is the product of its base and its altitude, and that of a triangle is equal to half that product.

The phrase 'the product of two lines,' or 'multiplying one line by another, is only an abridged manner of expressing the multiplication of the *number of parts* in one of the lines by the *number of parts* in the other. Multiplication is an operation which can only be effected, properly speaking, by a *number* and not by a *line*.

⁂ (187) Cor. 14.—The area of a square is found numerically by multiplying the number of equal parts in the side of the square by itself. Thus a square whose side is twelve inches contains in its area 144 square inches. Hence, in arithmetic, when a number is multiplied by itself the product is called its square. Thus 9, 16, 25, &c. are the *squares* of 3, 4, 5, &c.; and 3, 4, 5, &c. are called the *square roots* of the numbers 9, 16, 25, &c. Thus *square* and *square root* are correlative terms.

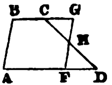
⁂ (188) Cor. 15.—*If the four sides of a quadrilateral figure* A B C D *be bisected, and the middle points* E F H G *of each pair of conterminous sides joined by right lines, those joining lines will form a parallelogram* E F H G *whose area is equal to half that of the quadrilateral.*



Draw C A and B D. The lines E F and G H are parallel to C A (179), and equal to half of C A (182). Therefore E F and G H are equal and parallel, and therefore (XXIII) E F H G is a parallelogram. But E B F is one-fourth of C B A, and G H D one-fourth of C D A (181), and therefore E B F and G D H are together one-fourth of the whole figure. In like manner E C G and F A H are together one-fourth of the whole, and therefore F B E, E C G, G D H, and H A F are together one-half of the whole figure, and therefore E F H G is equal to half the figure.

⁂ (189) Cor. 16.—*A trapezium is equal to a parallelogram in the same altitude, and whose base is half the sum of the parallel bases.*

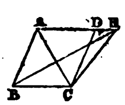
Let C D be bisected at H, and through H draw G F parallel to A B.



Since C G and F D are parallel, the angles G C H and G are respectively equal to D, and H F D (XXIX) and C H is equal to H D, therefore (XXVI) C G is equal to F D, and the triangle C H G to the triangle D H F. Therefore A F and B G are together equal to A D and B C, and the parallelogram A G to the trapezium A C; and since A F and B G are equal, A F is half the sum of A D and B C.

Proposition XLI. Theorem.

(190) If a parallelogram (B D) and a triangle (B E C) have the same base and be between the same parallels, the parallelogram is double of the triangle.



Draw C A. The triangle B E C is equal to the triangle B A C (XXXVII); but BD is double of the triangle B A C (XXXIV), therefore B D is also double of the triangle B E C.

(191) This proposition may be generalized thus: *If a parallelogram and triangle have equal bases and altitudes, the parallelogram is double the triangle* (175).

(192) Also, *If a parallelogram and a triangle have equal altitudes, and the base of the triangle be double the base of the parallelogram, the parallelogram and triangle will be equal.*

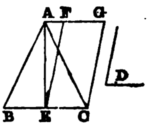
(193) *If a parallelogram and triangle have equal bases, and the altitude of the triangle be double the altitude of the parallelogram, they will be equal.*

PROPOSITION XLII. PROBLEM.

(194) To construct a parallelogram equal to a given triangle (B A C) and having an angle equal to a given one (D).

Solution.

Through the point A draw the right line A F parallel to B C, bisect B C the base of the triangle in E, and at the point E and with the right line C E make the angle C E F equal to the given one D; through C draw C G parallel to E F until it meet the line A F in G. C F is the required parallelogram.



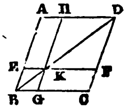
Demonstration.

Because E C is parallel to A G (const.), and E F parallel to C G, E G is a parallelogram, and has the angle C E F equal to the given one D (const.); and it is equal to the triangle B A C, because it is between the same parallels and on half of the base of the triangle (192).

PROPOSITION XLIII. THEOREM.

(195) In a parallelogram (A C) the complements (A K and K C) of the parallelograms about the diagonal (E G and H F) are equal.

Draw the diagonal B D, and through any point in it K draw the right lines F E and G H parallel to B C and B A; then E G and H F are the parallelograms about the diagonal, and A K and K C their complements.



Because the triangles B A D and B C D are equal (XXXIV), and the triangles B G K, K F D are equal to B E K, K H D (XXXIV); take away the equals B G K and K E B, D F K and K H D from the equals B A D and B C D, and the remainders, namely, the complements A K and K C, are equal.

(196) Each parallelogram about the diagonal of a lozenge is itself a lozenge equiangular with the whole. For since A B and A D are equal, A B D and A D B are equal (V). But E K B and A D B are equal (XXIX), therefore E K B and E B K are equal, therefore E K and E B are equal, and therefore E G is a lozenge. It is evidently equiangular with the whole.

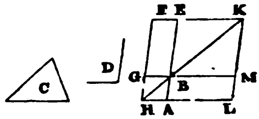
(197) It is evident that the parallelograms about the diagonal, and also their complements, are equiangular with the whole parallelogram; for each has an angle in common with it (152).

PROPOSITION XLIV. PROBLEM.

(198) To a given right line (A B) to apply a parallelogram which shall be equal to a given triangle (C), and have one of its angles equal to a given angle (D).

Solution.

Construct the parallelogram B E F G equal to the given triangle C, and having the angle B equal to D, and so that B E be in the same right line with A B; and produce F G, and through A draw A H parallel to B G, and join H B. Then because H L and F K are parallel the angles L H F and F are together equal to two right angles, and therefore B H F and F are together less than two right angles, and therefore H B and F E being produced will meet as at K. Produce H A and G B to meet K L parallel to H F, and the parallelogram A M will be that which is required.



Demonstration.

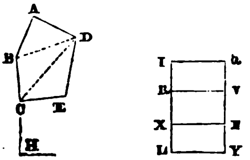
It is evidently constructed on the given line A B; also in the parallelogram F L, the parallelograms A M and G E are equal (XLIII); but G E is equal to C (const.), therefore A M is equal to C. The angle E B G is equal to A B M (XV), but also to D (const.), therefore A B M is equal to D. Hence A M is the parallelogram required.

PROPOSITION XLV. PROBLEM.

(199) To construct a parallelogram equal to a given rectilinear figure A B C E D, and having an angle equal to a given one.

Solution.

Resolve the given rectilinear figure into triangles; construct a parallelogram R Q equal to the triangle B D A (XLIV), and having an angle I equal to the given angle H; on a side of it, R V, construct the parallelogram X V equal to the triangle C B D, and having an angle equal to the given one (XLIV), and so on construct parallelograms equal to the several triangles into which the figure is resolved. L Q is a parallelogram equal to the given rectilinear figure, and having an angle I equal to the given angle H.



Demonstration.

Because R V and I Q are parallel the angle V R I together with I is equal to two right angles (XXIX); but V R X is equal to I (const.), therefore V R I with V R X is equal to two right angles, and therefore I R and R X form one right line (XIV); in the same manner it can be demonstrated, that R X and X L form one right line, therefore I L is a right line, and because Q V is parallel to I R the angle Q V R together with V R I is equal to two right angles (XXIX); but I R is parallel to V F, and therefore I R V is equal to F V R (XXIX), and therefore Q V R together with F V R is equal to two right angles, and Q V and F V form one right line (XIV); in the same manner it can be demonstrated of V F and F Y, therefore Q Y is a right line and also is parallel to I L; and because L Y and R V are parallel to the same line X F, L Y is parallel to R V (XXX); but I Q and R V are parallel, therefore L Y is parallel to I Q, and therefore L Q is a parallelogram, and it has the angle I equal to the given angle H, and is equal to the given rectilinear figure A B C E D.

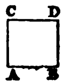
(200) Cor.—Hence a parallelogram can be applied to a given right line and in a given angle equal to a given rectilinear figure, by applying to the given line a parallelogram equal to the first triangle.

Proposition XLVI. Problem.

(201) On a given right line (A B) to describe a square.

Solution.

From either extremity of the given line A B draw a line A C perpendicular (XI), and equal to it (III); through C draw C D parallel to A B (XXXI), and through B draw B D parallel to A C; A D is the required square.



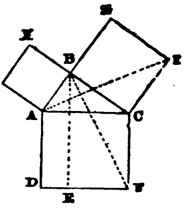
Demonstration.

Because A D is a parallelogram (const.), and the angle A a right angle, the angles C, D, and B are also right (153); and because A C is equal to A B (const.), and the sides C D and D B are equal to A B and A C (XXXIV), the four sides A B, A C, C D, D B are equal, therefore A D is a square.

Proposition XLVII. Theorem.

(202) In a right angled triangle (A B C) the square of the hypotenuse (A C) is equal to the sum of the squares of the sides (A B and C B).

On the sides A B, A C, and B C describe the squares A X, A F, and B I, and draw B E parallel to either C F or A D, and join B F and A I.

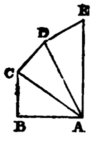


Because the angles I C B and A C F are equal, if B C A be added to both, the angles I C A and B C F are equal, and the sides I C, C A are equal to the sides B C, C F, therefore the triangles I C A and B C F are equal (IV); but A Z is parallel to C I, therefore the parallelogram C Z is double of the triangle I C A, as they are upon the same base C I, and between the same parallels (XLI); and the parallelogram C E is double of the triangle B C F, as they are upon the same base C F, and between the same parallels (XLI); therefore the parallelograms C Z and C E, being double of the equal triangles I C A and B C F, are equal to one another. In the same manner it can be demonstrated, that A X and A E are equal, therefore the whole D A C F is equal to the sum of C Z and A X.

⁂ (203) Cor. 1.—Hence if the sides of a right angled triangle be given in numbers, its hypotenuse may be found; for let the squares of the sides be added together, and the square root of their sum will be the hypotenuse (187).

⁂ (204) Cor. 2.—If the hypotenuse and one side be given in numbers, the other side may be found; for let the square of the side be subtracted from that of the hypotenuse, and the remainder is equal to the square of the other side. The square root of this remainder will therefore be equal to the other side.

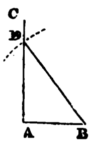
(205) Cor. 3.—Given any number of right lines, to find a line whose square is equal to the sum of their squares. Draw two lines A B and B C at right angles, and equal to the first two of the given lines, and draw A C. Draw C D equal to the third and perpendicular to A C, and draw A D. Draw D E equal to the fourth and perpendicular to A E, and draw A E, and so on. The square of the line A E will be equal to the sum of the squares of A B, B C, C D, &c., which are respectively equal to the given lines.



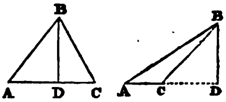
For the sum of the squares of A B and B C is equal to the square of A C. The sum of the squares of A C and C D, or the sum of the squares of A B, B C, C D is equal to the square of A D, and so on; the sum of the squares of all the lines is equal to the square of A E.

(206) Cor. 4.—To find a right line whose square is equal to the difference of the squares of two given right lines.

Through one extremity A of the lesser line A B draw an indefinite perpendicular A C; and from the other extremity B inflect on A C a line equal to the greater of the given lines (60); which is always possible, since the line so inflected is greater than B A, which is the shortest line which can be drawn from B to A C. The square of the intercept A D will be equal to the difference of the squares of B D and B A, or of the given lines.



(207) Cor. 3.—If a perpendicular (B D) be drawn from the vertex of a triangle to the base, the difference of the squares of the sides (A B and C B) is equal to the difference between the squares of the segments (A D and C D). For the square of A B is equal to the sum of the squares of A D and B D, and the square of C B is equal to the sum of the squares of C D and B D. The latter being taken from the former, the remainders, which are the difference of the squares of the sides A B and C B, and the difference of the squares of the segments A D and C D, are equal.

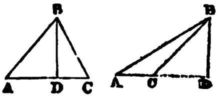


(208) To understand this corollary perfectly, it is necessary to attend to the meaning of the term *segments*. When a line is cut at any point, the intercepts between the point of section and its extremities are called its *segments*. When the point of section lies between the extremities of the line it is said to be cut *internally*; but when, as sometimes happens, it is not the line itself but its production that is cut, and therefore the point of section lies beyond one of its extremities, it is said to be cut *externally*. By due attention to the definition of *segments* given above, it will be perceived that when a line is cut *internally*, the line is the *sum* of its own *segments*; but when cut *externally*, it is their *difference*.

The case of a perpendicular from the vertex on the base of a triangle offers an example of both species of section. If the perpendicular fall within the triangle, the base is cut internally by it; but if it fall outside, it is cut externally. In both cases the preceding corollary applies, and is established by the same proof. The *segments* are in each case the intercepts A D and C D between the perpendicular and the extremities of the base.

(209) Cor. 4.—If a perpendicular be drawn from the vertex B to the base, the sums of the squares of the sides and alternate segments are equal.

For the sum of the squares of A B and B C is equal to the sum of the squares A B, B D and C D, since the square of B C is equal to the sum of the squares of B D and D C. For a similar reason, the sum of the squares of A B and B C is equal to the sum of the squares of A D, D B and B C. Hence the sum of the squares of A B, B D and D C is equal to that of A D, B D and B C. Taking the square of B D from both, the sum of the squares of A B and C D is equal to that of B C and A D.

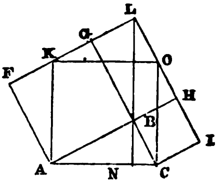


Whether we consider the 47th proposition with reference to the peculiar and beautiful relation established in it, or to its innumerable uses in every department of mathematical science, or to its fertility in the consequences derivable from it, it must certainly be esteemed the most celebrated and important in the whole of the elements, if not in the whole range of mathematical science. It is by the influence of this proposition, and that which establishes the similitude of equiangular triangles (in the sixth book), that Geometry has been brought under the dominion of Algebra, and it is upon these same principles that the whole science of Trigonometry is founded.

The XXXIId and XLVIIth propositions are said to have been discovered by Pythagoras, and extraordinary accounts are given of his exultation upon his first perception of their truth. It is however supposed by some that Pythagoras acquired a knowledge of them in Egypt, and was the first to make them known in Greece.

Besides the demonstration in the Elements there are others by which this celebrated proposition is sometimes established, and which, in a principle of such importance, it may be gratifying to the student to know.

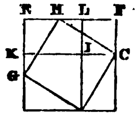
⁂ (210) 1° Having constructed squares on the sides A B, B C on opposite sides of them from the triangle, produce I H and F G to meet at L. Through A and C draw perpendiculars to the hypotenuse, and join K O.



In the triangles A F K and A B C, the angles F and B are equal, being both right, and F A K and B A C are equal, having a common complement K A B, and the sides F A and A B are equal. Hence A K and A C are equal, and in like manner C O and A C are equal. Hence A O is an equilateral parallelogram, and the angle at A being right, it is a square. The triangle L G B is, in every respect, equal to B C A, since B G is equal to B A, and L G is equal to B H or B C, and the angle at G is equal to the right angle B. Hence it is also equal in every respect to the triangle K F A. Since, then, the angles G L B and F K A are equal, K A is parallel to B L, and therefore A L is a parallelogram. The square A G and the parallelogram A L are equal, being on the same base A B, and between the same parallels (XXXV); and for the same reason the parallelograms A L and K N are equal, A K being their common base. Therefore the square A G is equal to the parallelogram K N.

In like manner the square C H is equal to the parallelogram O N, and therefore the squares A G and C H are together equal to A O.

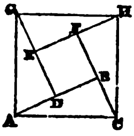
⁂ (211) 2° Draw A G perpendicular and equal to A C, and produce B A, and draw G D perpendicular to it. In the same manner draw C H perpendicular and equal to C A, and produce B C and draw H F perpendicular to it. Produce F H and D G to meet in E, and draw G H.



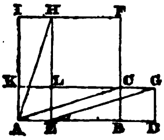
The triangles G D A and H F C are equal in every respect to A B C (XXVI). Hence F C, G D and A B are equal, and also H F, D A and B C, and the angles in each triangle opposed to these equal sides are equal. Also, since G A and H C are equal to A C, and therefore to each other, and the angles at A and C are right, A H is a square (XXXIII). Since G H is equal to A C, and the angles at G and H are right, it follows that the triangle G E H is in all respects equal to A B C (XXVI), in the same manner as for the triangles G D A and H F C.

Through C and A draw the lines C K and A L parallel to B D and B F. Since C B and A I are equal and also C B and A D, it follows that A K is the square of B C, and in like manner that C L is the square of A B. The parallelograms B I and K L have bases and altitudes equal to those of the triangle A B C, and are therefore each equal to twice the triangle, and together equal to four times the triangle. Hence B I and K L are together equal to A B C, C F H, H E G and G D A together. Taking the former and the latter successively from the whole figure, the remainders are in the one case the squares D I and C L of the sides B C and B A, and in the latter the square A H of the hypotenuse. Therefore, &c.

(212) 3° On the hypotenuse A C construct the square A H, and draw G D and H E parallel to C B and A B, and produce these lines to meet in F, E and D. The triangles A B C, A D G, G E H and H F C are proved in every respect equal (XXVI). It is evident, that the angles D, E, F, B are all right. But also since D G and A B are equal, and also G E and A D, taking the latter from the former D E and D B remain equal. Hence B E is a square on the difference D B of the sides; and therefore the square of A C is divided into four triangles, in all respects equal to A B C and the square B E of the difference of the sides.



Now let squares B G and B I be constructed on the sides, and take A E on the greater side equal to B C the less, and draw E H parallel to B C, and produce G C to K. Draw G E and A H.

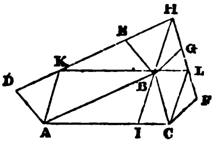


The part B E is the difference of the sides A B and B C. And since B F is equal to A B, F C is also the difference of the sides, wherefore F L is the square of this difference. Also since A E and B D are equal A B and D E are equal, therefore the parallelogram D L is double the triangle A B C. The sides and angles of the parallelogram A H are equal respectively to those of D L, and therefore these two parallelograms together are equal to four times the triangle A B C. Hence the squares A F and B G may be divided into four triangles G D E, G L E, A E H and A I H in all respects equal to the triangle A B C, and the square C H of the difference of the sides. But by the former construction the square of the hypotenuse was shown to be divisible into the same parts. Therefore, &c.

The peculiarity of this proof is, that it shows that the squares of the sides may be so dissected that they may be laid upon the square of the hypotenuse so as exactly to cover it, and *vice versa*, that the square of the hypotenuse may be so dissected as to exactly cover the squares of the sides.

(213) The forty-seventh proposition is included as a case of the following more general one taken from the mathematical collections of *Pappus*, an eminent Greek Geometer of the fourth century.

In any triangle (A B C) *parallelograms* A E *and* C G *being described on the sides, and their sides* D E *and* F G *being produced to meet at* H, *and* H B I *being drawn, the parallelogram on* A C *whose sides are equal and parallel to* B H *is equal to* A E *and* C G *together.*



For draw A K and C L parallel to B H, to meet D H and F H in K and L. Since A H is a parallelogram, A K is equal to B H, and for a similar reason C L is equal to B H. Hence C L and A K are equal and parallel, and therefore (XXXIII) A L is a parallelogram. The parallelograms A E and A H are equal, being on the same base A B, and between the same parallels, and also A H and K I whose common base is A K. Hence the parallelograms A E and K I are equal. In like manner the parallelograms C G and L I are equal, and therefore A E and C G are together equal to A L.

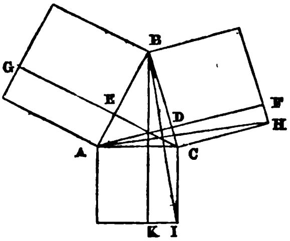
This proof is applied to the forty-seventh in (210).

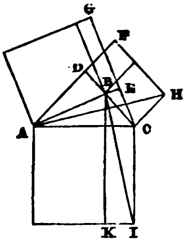
(214) The forty-seventh proposition is also a particular case of the following more general one:

In any triangle (A B C) *squares being constructed on the sides* (A B *and* B C) *and on the base; and perpendiculars* (A D F *and* C E G) *being drawn from the extremities of the base to the sides, the parallelograms* A G *and* C F *formed by the segments* C D, A E, *with the sides of the squares, will be together equal to the square of the base* A C.

For draw A H and B I; and also B K perpendicular to A C.

The parallelograms K C and C F are proved equal, exactly as C E and C Z are proved equal in the demonstration of the XLVIIth. And in like manner it follows, that A K and A G are equal, and therefore the square on A C is equal to the parallelograms A G and C F together.





If the triangle be right angled at B, the lines G E and D F will coincide with the sides of the squares, and the proposition will become the XLVIIth.

(215) If B be acute the perpendiculars A D and C E will fall within the triangle, and the parallelograms A G and C F are less than the squares of the sides; but if B be obtuse the perpendiculars fall outside the triangle, and the parallelograms A G and C F are greater than the squares of the sides.

Hence the forty-seventh proposition may be extended thus:

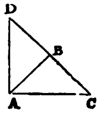
The square of the base of a triangle is less than, equal to, or greater than the sum of the squares of the sides, according as the vertical angle is less than, equal to, or greater than a right angle.

Proposition XLVIII. Theorem.

(216) If the square of one side (A C) of a triangle (A B C) be equal to the sum of the squares of the other two sides (A B and B C), the angle (A B C) opposite to that side is a right angle.

From the point B draw B D perpendicular (XI) to one of the sides A B, and equal to the other B C (III), and join A D.

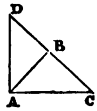
The square of A D is equal to the squares of A B and B D (XLVII), or to the squares of A B and B C which is equal to B D (const.); but the squares of



A B and B C are together equal to the square of A C (hyp.), therefore the squares of A D and A C are equal, and therefore the lines themselves are equal; but also D B and B C are equal, and the side A B is common to both triangles, therefore the triangles A B C and A B D are mutually equilateral, and therefore also mutually equiangular, and therefore the angle A B C is equal to the angle A B D; but A B D is a right angle, therefore A B C is also a right angle.

This proposition may be extended thus:

The vertical angle of a triangle is less than, equal to, or greater than a right angle, according as the square of the base is less than, equal to, or greater than the sum of the squares of the sides.



For from B draw B D perpendicular to A B and equal to B C, and join A D.

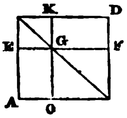
The square of A D is equal to the squares of A B and B D or B C. The line A C is less than, equal to, or greater than A D, according as the square of the line A C is less than, equal to, or greater than the squares of the sides A B and B C. But the angle B is less than, equal to, or greater than a right angle, according as the side A C is less than, equal to, or greater than A D (XXV, VIII); therefore, &c.

BOOK II.

DEFINITIONS.

(217) I. Every rectangle or right angled parallelogram is said to be contained by two right lines which contain one of its right angles.

(218) II. In any parallelogram either of the parallelograms about the diagonal (E K or O F) with the two complements (A G and G D) is called a gnomon.



(219) Next to the triangle, the most important rectilinear figure is the *rectangle* or right angled parallelogram. The areas of all figures whatever, whether bounded by straight lines or curves; are expressed by those of equivalent rectangles. To determine a rectangle it is only necessary to know two sides which are conterminous, for the other sides being opposed to these are equal to them, and the angles are all right. It is usual, therefore, to express a rectangle by its two conterminous sides, and it is said to be contained by these. Thus, if A and B express two lines which are the conterminous sides of a rectangle, the rectangle itself is called 'the rectangle under A and B.'

(220) It was proved in (186) that the area of a parallelogram can be expressed in numbers by multiplying the number which expresses the length of its base by that which expresses the length of its altitude. Hence, the area of a rectangle is expressed by multiplying the numbers representing its sides. The product then expresses the area. In arithmetic and algebra the product of two numbers is expressed by placing the sign × between them. Hence, we derive a shorter way of expressing a rectangle whose sides are A and B, *scil.* $A \times B$.

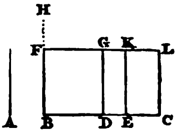
By what has been established in (186), it appears that the area of every parallelogram is expressed by the product of its base and altitude, and that every triangle is expressed by the product of any side and the perpendicular on it from the opposite angle.

The entire of the second book is appropriated to the investigation of the relations between the rectangles under the segments of right lines divided into two or more parts.

PROPOSITION I. THEOREM.

(221) If there be two right lines (A and B C), one of which is divided into any number of parts

(B D, D E, E C), the rectangle under the two lines is equal to the sum of the rectangles under the undivided line (A) and the several parts of the divided line (B C).



From the point B draw B H perpendicular to B C, take on it B F equal to A, and through F draw F L parallel to B C, and draw D G, E K, and C L parallel to B F.

It is evident that the rectangle B L is equal to the rectangles B G, D K, and E L; but the rectangle B L is the rectangle under A and B C, for B F is equal to A: and the rectangles B G, D K, and E L are the rectangles under A and B D, A and D E, and A and E C, for each of the lines B F, D G, and E K is equal to A (XXXIV, Book I.).

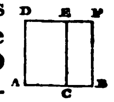
If the line B C be considered as the *sum* of the several lines B D, D E, &c. this proposition may be thus announced: 'The rectangles under one line and several others is equal to the rectangle under that line and the sum of the others.'

(222) COR.—*The rectangle under any two lines is equal to twice the rectangle under either of them and half the other, to three times the rectangle under either of them and a third of the other, &c. &c.*

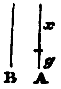
PROPOSITION II. THEOREM.

(223) If a right line (A B) be divided into any two parts (in C), the square of the whole line is equal to the sum of the rectangles under the whole (A B) and each of the parts (A C, C B).

On A B describe the square A D F B (XLV, Book I.), and through C draw C E parallel to A D. The square A F is equal to the rectangles A E and C F. But the rectangle A E is the rectangle under A B and A C, because A D is equal to A B (const.), and the rectangle C F is the rectangle under A B and C B, because C E is equal to A B (XXXIV, Book I. and const.).



(224) In this and the succeeding propositions there is no necessity for the absolute construction of the rectangles to establish the relations they express. We shall, therefore, subjoin to each a second demonstration independent of any construction.



Let A be the right line divided into the parts x and y. We are to prove that the square of A is equal to the rectangles $A \times x$ and $A \times y$ taken together.

Let B be drawn equal to A. By (1[*]) the rectangle $B \times A$ is equal to the rectangles $B \times x$ and $B \times y$ taken together; that is, (since B is equal to A) to the rectangles $A \times x$ and $A \times y$ taken together.

As we shall frequently have occasion to express the equality of quantities, the language will be abridged by the use of the sign $=$. Thus, '$A = B$' means 'the line A is equal to the line B.'

(225) The second book is generally found to be one of the greatest difficulties which the student has to encounter in plane geometry. One of the causes of this (if not the only cause) is, the great variety of forms under which the same proposition may present itself. We cannot do any thing more calculated to remove this difficulty, than to show from whence this variety of forms arises. We have already stated that the object of most of the propositions of this book is, to determine the relations between the rectangles under the parts of *divided lines.* We shall first confine our attention to a finite right line divided into *two* parts.

In this case there are three lines to be considered; 1st, the whole line; 2nd, its greater part; 3rd, its lesser part; and in the present proposition the square of the first is compared with the rectangles under it and the second and third.

If, however, the two parts be considered as two independent lines, the whole line must be considered as their *sum.* Under this view the second proposition becomes, 'The square of the sum of any two lines is equal to the rectangles under the sum and each of them.'

Again, if the whole line A be considered as the greater of two given right lines and one of the parts x as the less, the other part y must be their difference. Thus the greater line is, in fact, supposed to be divided into two parts equal to the less and difference. Under this view, the second proposition assumes the form, '*The square of the greater of two lines is equal to the rectangle under those lines together with the rectangle under the greater and difference.*'

These, though apparently different from the second proposition, as announced in the text, are really the same, no other change being made than in the names given to the line and its parts. They should not, therefore, be denominated *corollaries,* as is sometimes the case.

If W express the whole line, and P, p its parts, the proposition as announced in the text may be expressed thus:

$$\text{The square of } W = W \times P + W \times p.$$

(The sign $+$ interposed between two magnitudes signifies their sum.)

If L, l express any two lines and S express their sum, the second method of announcing the proposition may be expressed thus:

$$\text{The square of } S = S \times L + S \times l.$$

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[*] When a reference is made to a proposition without any mention of a 'Book,' the present book is to be understood.

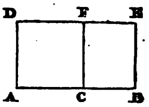
And if D represent the difference between L and l, the third method is:

$$\text{The square of } L = L \times l + L \times D$$

In the study of the second book considerable facility may be derived from the use of these symbols.

PROPOSITION III. THEOREM.

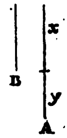
(226) If a right line (A B) be divided into any two parts (in C), the rectangle under the whole line (A B) and either part (A C) is equal to the square of that part (A C) together with the rectangle under the parts (A C and C B).



On A C describe the square A D F C, and through B draw B E parallel to A D, until it meet D F produced to E. The rectangle A E is equal to the square A D F C together with the rectangle C E.

But the rectangle A E is the rectangle under A C and A B, for A D is equal to A C (const.), and the square A D F C is the square of A C (const.), and the rectangle C E is the rectangle under A C and C B, for C F is equal to A C (const.).

Otherwise thus:



Let A be the right line divided into the parts x and y, and let B be another line equal to x. By (I) the rectangle $A \times B = B \times x + B \times y$. But since $B = x$, $\therefore$* the rectangle $B \times x$ is the square of x, and the rectangle $B \times y$ is equal to the rectangle $x \times y$. Hence, &c.

(227) Conformably to the observations on the last proposition, this may be announced in two other ways.

1. If the two parts of the divided line be considered as two independent lines, the whole line being their *sum*, the proposition becomes, 'The rectangle under the sum of two lines, and one of them, is equal to the square of that one together with the rectangle under the lines.'

2. If the whole line be considered as the greater, one part as the less, and the other as the difference, the proposition becomes, 'The rectangle under two lines is equal to the square of the less together with the rectangle under the less and difference.'

(228) COR. 1.—From this and the last proposition combined it follows, that *the difference of the squares of two lines is equal to the rectangle under their sum and difference.* For by the second, the

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[*] This sign expresses the word 'therefore.'

square of the greater is equal to the rectangle under the lines together with the rectangle under the greater and difference; and by the third, the rectangle under the lines is equal to the square of the less together with the rectangle under the less and difference. Hence, the square of the greater is equal to the square of the less together with the rectangles under the difference of the lines and the lines themselves respectively. But by (I) the rectangles under the difference, and the lines respectively, are together equal to the rectangle under the difference and the sum of the lines. Hence, the square of the greater of two lines is equal to the square of the less together with the rectangle under their sum and difference. This rectangle is therefore equal to the difference of their squares.

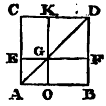
This, which is one of the most important principles established in the second book, is commonly deduced as a corollary from the fifth proposition. From the proof just given it appears, however, to be only a combination of the results of the second and third propositions.

(229) The second and third propositions might be incorporated and brought under one enunciation, thus: 'The difference between the rectangle under two lines and the square of one of them is the rectangle under that one and their difference.' If that one be the greater, this is the second proposition; and if it be the less, it is the third.

(230) COR. 2.—Since the greater of two lines is equal to the sum of the less and difference, it follows, that the sum of the lines is equal to twice the less together with the difference. Hence we may infer that the rectangle under the sum and difference is equal to the square of the difference together with the rectangle under twice the difference and less, (III), or to the square of the difference together with twice the rectangle under the difference and the less. Hence it follows, that *the difference of the squares of two lines exceeds the square of their difference by twice the rectangle under the less and difference.*

PROPOSITION IV. THEOREM.

(231) If a right line (A B) be divided into any two parts (in O), the square of the whole line is equal to the sum of the squares of the parts and twice the rectangle under the parts.



On A C describe the square A C D B, draw A D, and through O draw O K parallel to A C, cutting A D in G, and through G draw E F parallel to A B.

The square A C D B is equal to the squares E O and K F together with the rectangles C G and G B. But K F is the square of B O (196), and E O is the square of A O, for F G is equal to B O; and C G and G B together are equal to double the rectangle under the parts, because G K is equal to B O, and B G is the rectangle under the parts A O and O B, because O G and O A are equal (196).

Otherwise thus:

Let A be the line divided into the parts x and y. By (II) the square of A is equal to the rectangles A × x and A × y together; but by (III) the rectangle A × x is equal to the square of x together with the rectangle $x \times y$, and also the rectangle A × y is equal to the square of y together with the rectangle $x \times y$. Hence the square of A is equal to the squares of the parts x and y and twice the rectangle under them.

(232) Cor. 1.—*The square of a line is four times the square of its half.* For if the line be bisected, the squares of the parts are twice the square of half, and the rectangle under them is the square of the half.

(233) Cor. 2.—It appears, also, that *half the square of a line is equal to double the square of half the line.*

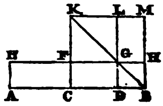
(234) This proposition may also be announced thus: 'The square of the sum of any two lines is equal to the sum of their squares together with twice the rectangle under them.'

(235) Cor. 3.—It will not be difficult to extend this proposition to a line divided into any number of parts; in this case *the square of the line will be equal to the sum of the squares of all the parts together with the double rectangle under every distinct pair of them.* Thus if it be divided into three parts x, y, z, the square of the whole line is equal to the sum of the squares of x, y, and z together with twice the sum of the rectangles $x \times y$, $y \times z$, and $x \times z$.

Proposition V. Theorem.

(236) **If a right line (A B) be cut into equal parts (in C), and into unequal parts (in D), the rectangle under the unequal parts (A D and D B), together with the square of the intermediate part (C D), is equal to the square of the half line (C B).**

On C B describe the square C K M B, draw K B, and through the point D draw D L parallel to C K, and cutting KB in G, and through G draw H G E parallel to A B, until it meet the line A E drawn through A parallel to C K.



Because the lines A C and C B are equal (const.), the rectangles AF and CH are equal (XXXVI, Book I.); but the rectangles C G and G M are also equal, therefore the rectangle A G is equal to the gnomon C H L (218); add to both the square F L, and the rectangle A G together with the square F L is equal to the square C K M B. But the rectangle A G is the rectangle under A D and D B, for D G is equal to D B, and F L is the square of C D, because F G and C D are equal (XXXIV, Book I.), and C K M B is the square of C B.

Otherwise thus:

The rectangle A D × D B is equal to the rectangles A C × D B and C D × D B, or to C B × D B and C D × D B (I). But the rectangle C B × D B is equal to the square of D B together with the rectangle C D × D B (III). Hence the rectangle A D × D B is equal to the square of D B together with twice the rectangle C D × D B. Add to both the square of C D, and the rectangle A D × D B together with the square of C D is equal to the squares of C D and D B together with twice the rectangle C D × D B, or to the square of C B (IV).

In this proposition the given finite line is supposed to be divided in two points, equally and unequally. In this case several distinct linear magnitudes are to be considered, viz. the whole line, the equal segments, the unequal segments, the *intermediate part*, or the part intercepted between the points of equal and unequal section.

(237) Between these several lines there are some obvious and important relations. The whole line is the *sum* of the unequal segments, and each of the equal segments is half the sum of the unequal segments. Again, since the greater segment exceeds the half line by the intermediate part, and the half line exceeds the lesser segment by the intermediate part, it follows, that the greater segment exceeds the lesser segment by twice the intermediate. Hence it appears, that the intermediate part is half the difference of the unequal parts.

(238) When three quantities are so related that the first exceeds the second by as much as the second exceeds the third, they are said to be in *arithmetical progression*. The first and third are called *extremes*, and the second is called the *mean*. The greater segment A D, the half line A C, and the lesser segment, are thus related, for A D exceeds A C by the intermediate C D; and again, A C or C B exceeds the lesser segment D B by the intermediate part C D. Hence the half line C A is an arithmetical mean between the unequal parts A D and D B.

(239) When three quantities are thus related, it appears therefore that the difference between the mean and each extreme is the same, and is therefore called the *common difference*. Thus the three lines A D, A C, D B are in arithmetical progression, the common difference being the intermediate part C D.

(240) It will be easy to establish similar conclusions, whatever be the nature of the quantities which are supposed to be in arithmetical progression; and it may in general be assumed, that ‘ the arithmetical mean is half the sum of the extremes, and that the common difference is half the difference of the extremes.’

(241) The fifth proposition may then be announced thus: ‘ The square of the arithmetical mean is equal to the rectangle under the extremes together with the square of the common difference.’

(242) If A D and D B be considered as two independent lines, the proposition assumes another form: ‘ The rectangle under any two lines together with the square of half their difference is equal to the square of half their sum.’

(243) Again, this proposition may still assume a different form. Let A C and C D be considered as two independent lines. The line A D will be their sum, and D B their difference. Thus the proposition becomes, ‘ The rectangle under the sum and difference of two lines toge-

ther with the square of the less is equal to the square of the greater:' or, 'The difference of the squares of two lines is equal to the rectangle under their sum and difference:' a result already obtained in (228).

(244) Cor. 1.—It appears that wherever the point of unequal section may be, the rectangle under the unequal parts together with the square of the intermediate make up the same sum; viz. the square of half the line. Hence it follows, that as the intermediate part diminishes the rectangle increases, and *vice versa.* If the point of unequal section be supposed continually to approach the middle point of the line, the rectangle will continually increase, since the intermediate continually diminishes; and when the point of unequal section arrives at the point of equal section, the rectangle under the unequal parts becomes equal to the rectangle under the equal parts, or to the square of half the line. If the point of unequal section be supposed to move beyond the middle point of the line, the rectangle begins to diminish. This affords a remarkable instance of a very extensive class of mathematical problems, in which the *maxima* or *minima* values of *variable* quantities are sought. In the present instance let us suppose a line given, and that it is required to cut it so that the rectangle under the segments shall be a *maximum;* that is, so that the rectangle under the segments shall be greater than the rectangle under any other segments into which the same line can be divided. Let us suppose that the point of section is first placed at the middle point of the line; the rectangle is then equal to the square of half the line. If it be moved toward either extremity, the rectangle will be diminished by the square of the space through which it is moved; and this diminution will continue until the point of section shall arrive at the extremity, when the square of the space through which it has been moved is the square of half the line, and the rectangle becomes absolutely nothing. Thus the rectangle is a *maximum* when the line is bisected, and its *maximum value* is the square of half the given line.

(245) Cor. 2.—Since the square of a divided line is composed of the sum of the squares of its parts and twice the rectangle under them, it follows, that the greater the rectangle is, the less will be the sum of the squares of the parts; and therefore, when the rectangle is a *maximum* the sum of the squares of the parts will be a *minimum.* Hence it appears, that the sum of the squares of the parts is a *minimum* when the line is bisected. The *minimum value* of the sum of the squares of the parts will evidently be twice the square of half the line.

(246) Cor. 3.—*Of all rectangles having the same perimeter the square contains the greatest area.* For by (242), the area of the square exceeds the area of any other isoperimetrical rectangle by the square of half the difference of the sides of the rectangle.

(247) Cor. 4.—*Of all rectangles equal in area the square is contained by the least perimeter.* For the square of the fourth part of the perimeter of a rectangle exceeds the area of an equivalent square by the square of half the difference of the sides of the rectangle; therefore, the perimeter of the rectangle must be greater than that of the equivalent square.

(248) We have already noticed the distinction between the *internal*

and *external* section of a line. In each case the segments are the parts intercepted between the point of section and the extremities (208). If a line be bisected and cut externally the lesser segment, the intercept between the points of equal and unequal section and the greater segment are in arithmetical progression, the common difference being half the given line. Hence it follows (240), that in this case the intermediate part between the two points of section is half the sum of the segments. This is a principle to which we shall have frequent occasion to refer.

(249) Cor. 5.—*If a perpendicular be drawn from the vertex of a triangle to the base, the rectangle under the sum and difference of the sides is equal to the rectangle under the sum and difference of the segments* (207).

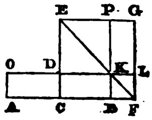
(250) Cor. 6.—*The difference between the squares of the sides of a triangle is equal to twice the rectangle under the base and the distance of the perpendicular from the middle point.*

For if the perpendicular fall within the base, this distance is half the difference of the segments, and the base is their sum; and if it fall outside the base, this distance is half the sum of the segments, and the base is their difference. Hence we may infer the principle from (207) and (243).

Proposition VI. Theorem.

(251) If a right line (A B) be bisected (in C), and produced to any point (F), the rectangle under the whole line, thus produced (A F), and the produced part (B F), together with the square of the half line (C B), is equal to the square of (C F) the line made up of the half and produced part.

On C F describe the square C E G F, draw E F, and through the point B draw B P parallel to F G, and cutting E F in K, through K draw L O parallel to C F and meeting A O, which is drawn through A parallel to C D.



Because A C and C B are equal (hyp.), the rectangle A D is equal to the rectangle C K (XXXVI, Book I.); but the rectangles C K and K G are equal (XLIII, Book I.), therefore A D is equal to K G; add to both C L, and A L is equal to the gnomon C L P; add to both D P, and the sum of A L and D P is equal to the square of C F. But A L is the rectangle under the whole produced line and the produced part, for F L is equal to B F (196), and D P is the square of the half C B, for it is the square of D K (196), and D K is equal to C B (XXXIV, Book I.).

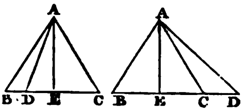
Otherwise thus:

The rectangle A F × F B is equal to the square of B F together with twice the rectangle C B × B F (I, III). Add to both the square of C B, and the rectangle A F × F B together with the square of C B is equal to the squares of C B and B F together with twice the rectangle under C B and B F, or (IV) to the square of C F.

(252) This proposition differs only in appearance from the fifth. In this case the line A B is cut externally at F, and the intermediate part C F is half the sum of the segments. We have shown that the fifth may be announced thus: 'The square of half the sum of two lines is equal to the rectangle under them together with the square of half their difference.' Now, in the present instance, C F is half the sum of A F and B F, and C B is half their difference, so that the present proposition is, in fact, identical with the fifth.

(253) Cor.—*If a line* A D *be drawn from the vertex* A *of an isosceles triangle to the base or its production, the difference between the squares of this line and the side of the triangle is the rectangle under the segments* B D × D C *of the base.*

For by (207) the difference of the squares of A D and A C is equal to the difference of the squares of the half base C E and the intermediate part D E; but this is equal to (V, VI) the rectangle B D × D C. If the line A D be perpendicular to the base it will coincide with A E, and the intermediate part D E will vanish.

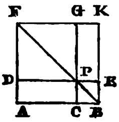


Proposition VII. Theorem.

(254) If a right line (A B) be divided into any two parts, the sum of the squares of the whole line (A B) and either segment (C B) is equal to double the rectangle under the whole, and that segment, together with the square of the other segment (A C).

Describe the square of A B, draw F B through the point C, draw C G parallel to A F, and through P, its intersection with F B, draw D E parallel to A B.

The square A K is equal to the rectangles A E and P K together with the square D G: add to both the square C E, and the squares A K and C E, taken together, are equal to the rectangles A E and C K together with the square D G.



But A E is equal to the rectangle under A B and C B, because C B and B E are equal (196), and C K is also equal to the rectangle under A B and C B, because K B is equal to A B (const.), and D G is the square of A C, because D P and A C are equal (XXXIV, Book I.).

Otherwise thus:

The square of A B is equal to the sum of the squares of A C and C B together with twice the rectangle A C × C B (IV). Add to both the square of C B, and the sum of the squares of A B and C B is equal to the square of A C together with twice the rectangle A C × C B and twice the square of C B. But the rectangle A C × C B together with the square of C B is equal to the rectangle A B × B C. Hence the sum of the squares of A B and B C is equal to twice the rectangle A B × B C together with the square of A C.

(255) Cor. 1.—If we consider A B and B C as two independent lines, and AC as their difference, this proposition will be thus announced: 'The sum of the squares of any two lines is equal to twice the rectangle under them together with the square of their difference.'

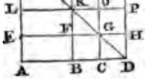
(256) Cor. 2.—Hence and by (IV) it appears, that *the square of the sum of two lines, the sum of their squares, and the square of their difference, are in arithmetical progression, the common difference being twice the rectangle under the lines.* For by (IV), the square of the sum exceeds the sum of the squares by twice the rectangle; and by (VII), the sum of the squares exceeds the square of the difference by twice the rectangle.

Proposition VIII. Theorem.

(257) If a right line (A C) be divided into any two parts (in B), the square of the sum of the whole line (A C), and either segment (B C), is equal to four times the rectangle under the whole line (A C), and that segment (B C), together with the square of the other segment (A B).

Produce A C till C D is equal to B C; on A D describe the square A R Z D (XLVI, Book I.), and through the points B and C draw B S and C V parallel to A R; having drawn R D, draw through the points G and K, E H and L P parallel to A D.

Because S V is equal to B C (XXXIV, Book I.), and B C to C D (const.), and C D to V Z, S V and V Z are equal, and therefore the rectangles S G and V H are equal (XXXVI, Book I.); but V H and A G are also equal (XLIII, Book I.), therefore S G is equal to A G; and because F G is equal to B C (XXXIV, Book I.), F G and C D are equal, and therefore the square F O is equal to the square C H; but also E K and K V are equal (XLIII, Book I.); to these equals, if the equals C H and F O be added, E K and C H



together shall be equal to S G, and therefore to A G: therefore A G, S G, and V H, together with E K and C H, are four times A G; but A G, S G, and V H, together with E K, C H, and the square L S, are equal to the square A Z; therefore A G, four times taken together with L S, is equal to A Z.

But A G is the rectangle under A C and B C, because C G is equal to C D (196), and therefore to B C, and L S is the square of A B, because A B and R S are equal (XXXIV, Book I).

Otherwise thus:

By (IV), the square of the sum of A C and B C is equal to the sum of their squares together with twice the rectangle under them; and by (VII), the sum of the squares of A C and B C is equal to twice the rectangle under them together with the square of A B; hence the square of the sum of A C and B C is equal to four times the rectangle under them together with the square of A B.

This proposition may evidently be expressed thus: 'The square of the sum of two lines is equal to four times the rectangle under them together with the square of their difference.'

Otherwise thus:

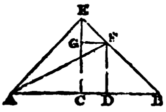
By (V), the square of half the sum of two lines is equal to the rectangle under them together with the square of half the difference. Therefore four times the square of half the sum (or the square of the sum (232)) is equal to four times the rectangle under the lines together with four times the square of half the difference (or the square of the difference).

PROPOSITION IX. THEOREM.

(258) If a right line be cut into equal parts (in C), and into unequal parts (in D), the sum of the squares of the unequal parts (A D and D B) is equal to double the sum of the squares of the half (A C) and of the intermediate part (C D).

From the point C draw C E perpendicular to A B and equal to either A C or C B (XI, III, Book I.), join A E and E B, and through D draw D F parallel to C E, and through F draw F G parallel to C D, and join F A.

Because the angle A C E is a right angle, and the sides A C and C E are equal (const.), C E A is half a right angle; in the same manner it can be demonstrated, that C E B is half a right angle, therefore A E B is a right angle; on account of the parallels G F and C D, the angle E G F is equal E C B (XXIX, Book I.), therefore E G F is a right angle; but G E F is half a right angle, therefore G F E is also half a right angle, and therefore G E and G F are equal (VI, Book I.); likewise F D B is



a right angle, because it is equal to the angle E C B, on account of the parallels F D and C E; but D B F is half a right angle, therefore D F B is half a right angle, and therefore D F and D B are equal (VI, Book I.). Since, therefore, A C and C E are equal and the angle A C E right, the square of A E is double the square of A C, and because E G and F G are equal and the angle E G F right, the square of E F is double the square of G F; but G F and C D are equal, therefore the square of E F is double the square of C D, and therefore the squares of A E and E F are double the squares of A C and C D; but because the angle A E F is right, the square of A F is equal to the squares of A E and E F (XLVII, Book I.), therefore the square of A F is double the squares of A C and C D; but the square of A F is equal to the squares of A D and D F as the angle A D F is right, therefore the sum of the squares of A D and D F is double the sum of the squares of A C and C D; but D F and D B are equal, and therefore the sum of the squares of A D and D B is double the sum of the squares of A C and C D.

Otherwise thus:

The square of A D is equal to the squares of A C and C D together with twice the rectangle A C × C D, or to the sum of the squares of B C and C D together with twice the rectangle B C × C D. Add to both the square of B D, and we have the sum of the squares of A D and D B equal to the sum of the squares of B C and C D together with twice the rectangle B C × C D and the square of B D. But twice this rectangle with the square of B D is equal to the sum of the squares of B C and C D, ∴ &c.

This proposition may be expressed (237) thus: 'The sum of the squares of any two lines is equal to twice the square of half their sum together with twice the square of half their difference.'

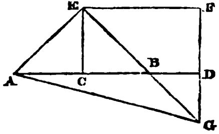
Otherwise thus:

By (256) the square of the sum, the sum of the squares, and the square of the difference are in arithmetical progression; ∴ the sum of the squares is equal to (240) half the square of the sum together with half the square of the difference, or to twice the square of half the sum together with twice the square of half the difference (233).

PROPOSITION X. THEOREM.

(259) If a right line (A B) be bisected (in C), and produced to any point (D), the square of the whole line thus produced (A D) together with the square of the produced part (B D), is equal to double the square of the line (C D) made up of the half and produced part together with double the square of (A C) half the given line.

From the point C draw C E perpendicular to A B and equal to either C A or C B; join A E, and draw through the point E the line E F parallel to A B, and through D, D F parallel to C E; and because the angles C E F and D F E are equal to two right angles, on account of the parallel lines C E and D F (XXIX, Book I.), the angles B E F and D F E are less than two right angles, therefore the lines E B and F D, if produced, shall meet: let them meet in G, and draw G A.



Because C A and C E are equal (const.), and the angle C a right angle (const.), the angle C E A is half a right angle, and in the same manner it is proved that C E B is half a right angle, therefore A E B is a right angle; and because D G and E C are parallel, the alternate angles G D B, E C B are equal, therefore G D B is a right angle; also the angles D B G and E B C are equal (XV, Book I.), but E B C is half a right angle, therefore D B G is half a right angle, and also D G B, and therefore the sides D B and D G are equal (VI, Book I.); and because E G F is half a right angle, and the angle at F right, being equal to its opposite C (XXXIV, Book I.), F E G is half a right angle, and therefore the sides E F and F G are equal.

Because A C and C E are equal, and the angle A C E right, the square of A E is double the square of A C, and because G F and F E are equal, and the angle F right, the square of G E is double the square of E F; but E F and C D are equal (XXXIV, Book I.), therefore the square G E is double the square of C D; the square of A E is also double the square of A C, therefore the squares of A E and E G are together double the squares of A C and C D: but the square of A G is equal to the squares of A E and E G, and is therefore double the squares of A C and C D, and the squares of A D and D G are equal to the square of A G, and therefore double the squares of A C and C D; but B D and D G are equal, and therefore the squares of A D and D B are double the squares of A C and C D.

This proposition is identical with the ninth, and the second and third demonstrations of the ninth may, without any change whatever, be applied to this. This proposition holds the same relation to the ninth as the sixth does to the fifth.

(260) These ten propositions contain the whole theory of the relations of the rectangles and squares of divided lines and their parts. All the relations which have been here established respecting lines may be applied to numbers, by supposing a number to be divided into parts equal or unequal, or both, as the case may be, and substituting the product of the parts in place of the rectangle under them. Thus, the fifth proposition, applied to numbers, is thus expressed: 'The product of two

numbers together with the square of half their difference is equal to the square of half their sum.' If, for example, the numbers be 6 and 10, the product is 60, the square of half their difference is 4, which added to 60 gives 64, which is the square of 8, or the half of 16.

When lines are expressed numerically, various problems may be proposed respecting them, the solution of which may be derived from the preceding propositions. We shall here subjoin some of these problems, which will probably be sufficient to familiarize the student with such investigations.

(261) *Given the sum and difference of two magnitudes to find the magnitudes themselves.*

Add half the difference to half the sum, and the result is the greater of the sought magnitudes, and subtract half the difference from half the sum, and the remainder is the less.

(262) Since the area of a rectangle is equal to the product of its sides, it follows that *if the area be divided by one side the quote will be the other side.* It is scarcely necessary to observe, that when we speak of the multiplication or division of geometrical magnitudes we mean only to apply these operations to these magnitudes expressed numerically.

(263) There are five quantities depending on a rectangle, any two of which being given, the sides of the rectangle can be found.

1° The sum of the sides.
2° The difference of the sides.
3° The area.
4° The sum of the squares of the sides.
5° The difference of the squares of the sides.

These five data being combined in pairs give the following ten problems.

(264) I. If 1° and 2° be given, the sides are found by (261).

(265) II. If 1° and 3° be given, subtract the area from the square of half the sum, and the remainder is the square of half the difference, which reduces the problem to I.

(266) III. If 1° and 4° be given, subtract the sum of the squares from the square of the sum and the remainder is twice the rectangle, which reduces the problem to II.

(267) IV. If 1° and 5° be given, divide the difference of the squares by the sum of the lines and the quote is their difference, which reduces the problem to I.

(268) V. If 2° and 3° be given, add the square of half the difference of the sides to the area and the result is the square of half the sum of the sides, which reduces the problem to I.

(269) VI. If 2° and 4° be given, subtract the square of the difference from the sum of the squares and the remainder is twice the rectangle, which reduces the problem to V.

(270) VII. If 2° and 5° be given, divide the difference of the squares by the difference of the sides and the quote is the sum of the sides, which reduces the problem to I.

(271) VIII. If 3° and 4° be given, add twice the rectangle to the

sum of the squares and the result is the square of the sum. Thus the problem is reduced to II.

(272) IX. If 4° and 5° be given, the squares of the sides may be found by (261), and thence the sides themselves.

(273) X. If 3° and 5° be given; this is the only case which presents any considerable difficulty. We shall postpone the investigation of this case until we shall have proceeded farther in this book, as it will require the aid of some principles which still remain to be established.

(274) It is evident that if two rectangles agree in any two of the five quantities expressed in (263), their sides will be equal. Thus, if their areas and the sums of their sides be equal the sides themselves will be equal; for if the equal areas be taken from the squares of half the sums of the sides, the remainders, which are the squares of the half differences of the sides, will be equal; and since the half differences and half sums of the sides are equal, it is evident that the sides themselves will be equal. In a similar way, the sides may be proved equal if the rectangles agree in any two of the five quantities.

(275) Hence also it appears, that *if two equal right lines be cut internally so that the rectangles under their segments be equal, the segments themselves are equal;* or, if the sums of the squares of the segments, or the differences of the squares of the segments, be equal, the segments themselves will be equal. The student will find no difficulty in proving these, and applying similar investigations to equal lines cut externally.

(276) If the three sides of a triangle be given in numbers, its area may be found. For, let the difference of the squares of any two unequal sides be found; half of this will be equal to the rectangle under the remaining side and the distance of the perpendicular on it from its middle point. If this half difference, therefore, be divided by the remaining side, the quote will be the distance of the perpendicular from the middle point. This quote, added to half the divisor, will give the greater segment made by the perpendicular. The square of this segment, subtracted from the square of the greater side, leaves a remainder equal to the square of the perpendicular; the square root of this remainder is the perpendicular itself, which multiplied into half the divisor gives the area of the triangle.

If it happen that the triangle is isosceles, the perpendicular is obtained by subtracting the square of half the base from the square of either of the equal sides, and taking the square root of the remainder. This multiplied by half the base gives the area.

(277) There are some well-known properties of a right angled triangle, which may be derived from the propositions of the second book, combined with the 47th proposition of the first book. It will not be necessary to trace the steps of each proof. Let S and S' be the sides about the right angle, H the hypotenuse, P the perpendicular, s and s' the segments of the hypotenuse, conterminous with S and S' respectively.

1. The square of $P = s \times s'$. For, by (XLVII, Book I.), the squares

of S and S' together are equal to the squares of s, s', and twice the square of P. But, by (IV), the square of H is equal to the squares of s and s' together with twice the rectangle $s \times s'$. Hence, &c.

2. The square of $S = H \times s$. For the square of S = the squares of P and s; but the square of $P = s \times s'$ $\because$ by (III), &c.

3. In like manner the square of $S' = H \times s'$.

4. $H \times P = S \times S'$, for each is twice the area (186).

(278) The converses of these properties may be easily established, *scil.* that a triangle, having any of these properties, must be right angled.

1. If the square of $P = s \times s'$, let twice that square be added to the sum of the squares of s and s', and we shall, by (XLVII, Book I.), obtain a magnitude equal to the sum of the squares of S and S'; and, since the square of P is equal to twice the rectangle $s \times s'$, we shall also have the same magnitude (IV) equal to the square of H. Hence, by (XLVIII, Book I.), the angle opposite to H is right.

2. If the square of $S = H \times s$, we have also the square of S = the squares of P and s. Take the square of s from both, and we have the square of $P = s \times s'$; therefore, by the last case, the angle opposite to H is right.

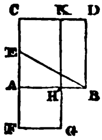
In these cases, the perpendicular P is supposed to fall within the side H; if not, the propositions are not necessarily true.

3. If $H \times P = S \times S$. In this case $S \times S'$ is twice the area, and also S × the perpendicular on it from the opposite angle is twice the area; $\because$ S' is equal to that perpendicular, and therefore must be the perpendicular itself, since no line equal to it could be drawn from the same point.

PROPOSITION XI. PROBLEM.

(279) To divide a given finite right line (A B) so that the rectangle under the whole line and one segment shall be equal to the square of the other segment.

From the point A erect A C perpendicular and equal to the given line A B, bisect it in E, join E B, produce C A until E F is equal to E B, and in the given line A B take A H equal to A F; the square of A H is equal to the rectangle under the other segment H B and the whole line A B.



Complete the square of A B, draw through H the right line G K parallel to A C, and through F the right line F G parallel to A B.

Because C A is bisected in E, and A F is added to it, the rectangle under C F and F A together with the square of E A is equal to the square of E F (VI), or to the square of E B which is equal to E F (const.), and therefore to the squares of E A and A B (XLVII, Book I.); take away the common square of E A, and the rectangle under C F and F A is equal to the square of A B: but because A F and F G are equal, C G is the rectangle under C F and F A, therefore C G and A D are equal, and if the common rectangle C H be taken away A G and H D are equal; but A G is the square of A H, for A H and A F are equal (const.), and the angle A is a right angle, and H D is the rectangle under A B and H B, for B D is equal to A B.

A line divided, as in this proposition, is said (vide Book VI.) to be cut 'in extreme and mean ratio.'

(280) Cor. 1.—By attending to the solution of this problem, it will appear that, in order to cut a line in extreme and mean ratio, it is first necessary to produce it in extreme and mean ratio; that is, to produce it so that the rectangle under the whole produced line and produced part shall be equal to the square of the line itself. In the demonstration of the proposition, it appears that the rectangle C F × F A is equal to the square of C A, and therefore C A has been produced to F in this way, and C A is equal to the given line A B.

(281) Cor. 2.—Considering C F as a line cut in extreme and mean ratio at A, it will easily appear that the rectangle under the greater segment, and the difference of the segments, is equal to the square of the lesser segment; for A C is the greater segment, and is equal to A B, A F, which is equal to A H, is the less, and therefore H B is the difference of the segments. But by the demonstration of the proposition A B × H B is equal to the square of A H.

Hence it appears, that *if a line be cut in extreme and mean ratio, the greater segment will be cut in the same manner, by taking on it a part equal to the less. And the less will be similarly cut, by taking on it a part equal to the difference, and so on.*

(282) We have here taken for granted that if the rectangle C F × F A = the square of C A, that C A is greater than A F. This is, in fact, also taken for granted in the demonstration of the proposition itself. It is, however, easily proved. The rectangle C F × F A is equal to the rectangle C A × A F, together with the square of A F, ∴ the square of C A exceeds the square of A F by the rectangle C A × A F, and ∴ the line C A must be greater than A F.

(283) Cor. 3.—Hence it also appears, that when a line is cut in extreme and mean ratio, the rectangle under its segments is equal to the difference between their squares.

Let A be a line cut in extreme and mean ratio, and G its greater segment, L its lesser segment, D the difference of its segments. The student will find no difficulty in establishing the following properties.

(284) 1. The sum of the squares of A and L is equal to three times the square of G.

(285) 2. The square of the sum of A and L is equal to five times the square of G.

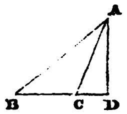
(286) 3. A × D = G × L.

(287) 4. The square of L = G × D.

It may also be shown, that a line cut so as to have any of these properties will be cut in extreme and mean ratio.

PROPOSITION XII. THEOREM.

(288) In any obtuse angled triangle (A B C), the square of the side (A B) subtending the obtuse angle exceeds the sum of the squares of the sides (B C and C A) which contain the obtuse angle by double the rectangle under either of these sides (B C), and the external segment (C D) between the obtuse angle and the perpendicular drawn from the opposite angle.

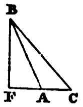


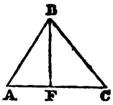
The square of B A is equal to the sum of the squares of A D and D B (XLVII, Book I.); but the square of D B is equal to the squares of D C and C B together with double the rectangle under D C and C B (IV); therefore the square of A B is equal to the squares of A D, D C, and C B together with double the rectangle under D C and C B; but the square of A C is equal to the squares of A D and D C (XLVII, Book I.); and therefore the square of A B is equal to the squares of A C and C B together with double the rectangle under B C and C D, therefore the square of A B exceeds the sum of the squares of A C and C B by double the rectangle under D C and C B.

It is evident that if the perpendicular were drawn from B to A C produced, it would in like manner be proved that double the rectangle under A C and its production would be equal to the excess of the square of A B above the squares of A B and B C. And hence it follows, that the rectangle B C × C D is equal to the rectangle under A C and its produced part.

PROPOSITION XIII. THEOREM.

(289) In any triangle (A B C) the square of the side (A B) subtending an acute angle (C) is less than the sum of the squares of the sides (A C and C B) containing that angle, by twice the rectangle under either of them (A C) and the segment between the acute angle and the perpendicular (B F), let fall from the opposite angle.





The squares of A C and C F are equal to twice the rectangle under A C and C F together with the square of A F (VII), and if the square of the perpendicular B F be added to both, the squares of A C, C F, and B F are equal to twice the rectangle under A C and C F together with the squares of B F and A F, or with the square of A B, which is equal to them; but the squares of B F and C F are equal to the square of B C, and therefore the squares of B C and A C are equal to twice the rectangle under A C and C F together with the square of A B; therefore the square of A B is less than the sum of the squares of A C and C B by twice the rectangle under A C and C F.

If the angle A happen to be a right angle, the perpendicular B F will coincide with B A, and the points F and A will be the same, but the demonstration remains unchanged.

(290) If the angle A be right the double rectangle A C × C F becomes equal to twice the square of A C, and the proposition becomes equivalent to the forty-seventh of the first book.

(291) This proposition and the twelfth may be reduced to one, thus: 'The difference between the square of one side of a triangle, and the sum of the squares of the other two sides, is equal to twice the rectangle under either of these two sides and the intercept between the perpendicular on it and the angle included by the sides.'

(292) Cor. 1.—*If a perpendicular to* B C *be drawn from the angle* A, *the rectangle under the side* B C *and the part intercepted between this perpendicular and* C *is equal to the rectangle* A C × C F. For each of these rectangles is half the difference between the square of A B and the squares of B C and A C.

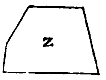
(293) Cor. 2.—If the three sides of a triangle be given in numbers, its area may be found by these principles. Find half the difference between the square of any side and the sum of the squares of the other two sides. This is the rectangle under either of those other two sides and the intercept between the perpendicular and the included angle Let this then be divided by either of the other sides, and the quote will be that intercept. Take its square from the square of the other side, and the remainder is the square of the perpendicular, the square root of which is the perpendicular itself. This multiplied by half the divisor gives the area.

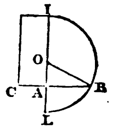
If it happen that the square of one side be equal to the sum of the squares of the other two, the angle included by those two must be right, and in that case the area may at once be found by taking half their product.

PROPOSITION XIV. THEOREM.

(294) To construct a square equal to a given rectilinear figure (Z).

Construct a rectangle C I equal to the given rectilinear figure (XLV, Book I.); if the adjacent sides be equal, the problem is solved.





If not, produce either side I A, and make the produced part A L equal to the adjacent side A C; bisect I L in O, and from the centre O with the radius O L describe a semicircle L B I, and produce C A till it meet the periphery in B; the square of A B is equal to the given rectilinear figure.

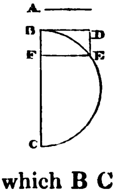
For draw O B, and because I L is bisected in O and cut unequally in A, the rectangle under I A and A L together with the square of O A is equal to the square of O L (V), or of O B, which is equal to O L, and therefore to the squares of O A and A B (XLVII, Book I.); take away from both the square of O A, and the rectangle under I A and A L is equal to the square of A B; but the rectangle under I A and A L is equal to I C, for A L and A C are equal (const.); therefore the square of A B is equal to the rectangle I C, and therefore to the given rectilinear figure Z.

(295) Schol. From this proposition it appears, that if a perpendicular B A be drawn from any point in a semicircle to the diameter, the square of the perpendicular is equal to the rectangle under the segments into which it divides the diameter.

(296) The following is a selection from some of the most useful and remarkable theorems and problems which may be inferred from the second book.

(297) *To divide a line internally so that the rectangle under its segments shall have a given magnitude.*

Let the given magnitude be equal to the square of the line A, and let B C be the given line.

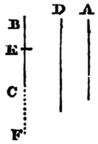


On B C describe a semicircle, and through B draw B D perpendicular to B C and equal to A. Draw D E parallel to B D and E F perpendicular to B C. Then B C is cut as required at F. This appears from (295).

It is evident that if A were greater than half of B C, the parallel D E would not meet the semicircle, and the problem would be impossible; and since, in general, the parallel meets the circle at two points, there are two points at which B C may be cut as required, and these points are at equal distances from its middle point.

(298) *To cut a line externally so that the rectangle under the segments shall be equal to a given magnitude.*

Let the given magnitude be equal to the square of A, and find a line D whose square is equal to the sum of the squares of A and half the given line B C. From the middle point E of B C take E F equal to this line, and F is the point of external section sought. This is evident from (VI).

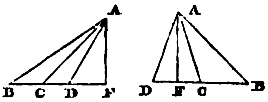


Since E F may be taken from the middle point towards either extremity, there are two points of section which solve the problem, equally distant from the middle point.

(299) *If a line* A C *be drawn from the vertex* A *of a triangle to the middle point* C *of the opposite side, the sum of the squares of the other sides* B A *and* A D *is equal to twice the sum of the squares of the bisector* A C *and half* B C *of the bisected side.*

If A B = A D then A C B is a right angle, and the proposition is evident by XLVII, Book I.

If not, draw the perpendicular A F.

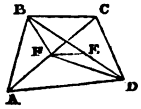


By (XII), the square of A B exceeds the sum of the squares of A C and C B by twice B C × C F, or twice D C × C F.

By (XIII), the sum of the squares of A C and C D, or A C and C B, exceeds the square of A D by twice C D × C F. Hence it appears, that the sum of the squares of the bisector A C and half the base is an arithmetical mean between the squares of the sides A C, A D; and therefore (240) the sum of the squares of the sides is equal to twice the sum of the squares of the bisector and half the bisected side.

(300) *The sum of the squares of the sides of a quadrilateral figure* A B C D, *is equal to the sum of the squares of the diagonals together with four times the square of the line* E F *joining their points of bisection.*

Draw B F and D F. The sum of the squares of A B and B C is equal to twice that of B F and C F, and the sum of the squares of A D and D C is equal to twice that of D F and C F (299). But also the sum of the squares of B F and D F is equal to twice that of E F and D E. Hence the proposition is manifest.



(301) *The sum of the squares of the sides of a parallelogram is equal to that of the diagonals.*

For in that case the line E F vanishes, since the diagonals bisect each other (155).

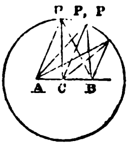
(302) *If the sum of the squares of the sides of a quadrilateral figure be equal to the sum of the squares of the diagonals, the quadrilateral will be a parallelogram.*

For otherwise it would be greater by four times the square of the line E F.

(303) *If lines be drawn from the three angles of a triangle to the middle points of the opposite sides, three times the sum of the squares of the sides is equal to four times that of the bisectors.*

Let A, B, C, be the sides and a, b, c the corresponding bisectors. The sum of the squares of B and C is equal twice the sum of the squares of a and half of A, or twice the sum of the squares of B and C is equal to four times the square of a together with the square of A. In like manner twice the sum of the squares of A and B is equal to four times the square of c together with the square of C, and twice the sum of the squares of A and C is equal to four times the square of b together with the square of B. Hence, by adding these equals, and taking the sum of the squares of the sides from both, the proposition follows.

(304) *If with the middle point* C *of a finite right line* A B *as centre a circle be described, the sums of the squares of the distances of all points in this circle from the extremities of the right line are the same, and equal to twice the sum of the squares of the radius and half the given line.*



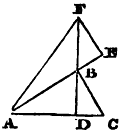
For the triangles A P B have a common base A B, and the bisectors C P of the base are equal, being radii of the circle. Hence the proposition follows from (299).

(305) Hence, if the base of a triangle and the sum of the squares of the sides be given, the locus of the vertex is a circle whose centre is the middle point of the base, and the square of whose radius is half the difference between the square of the base and the sum of the squares of the sides.

(306) *If a point be assumed within or without a rectangle, the sum of the squares of lines drawn from it to two opposite angles is equal to the sum of the squares of the lines drawn to the other two opposite angles.*

This is evident from (299), by considering that the diagonals are equal and bisect each other.

(307) In a right angled triangle A B C if a perpendicular B D be drawn, the rectangle A B × D C = the rectangle B D × B C. This might be easily derived from the third book, and still more simply from the sixth book. We shall in the present instance however, prove it by the 12th proposition of the second book.

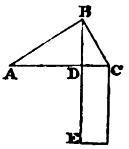


Produce A B and D B so that B E = D C, and B F = B C, and draw F E. The triangle B F E is equal in every respect to B C D, ∵ E is a right angle. Draw A F. Since E and D are right angles, the rectangle A B × B E = F B × B D (288 Obs.). But F B = B C and B E = D C, ∵ A B × D C = B C × B D.

(308) We shall now solve the tenth case of the class of problems mentioned in (273).

Given the difference of the squares of two lines and the rectangle under them to find the lines.

Let a line D C be found (XIV), whose square is equal to the given difference of squares, and on it let a rectangle C E be constructed equal to the given rectangle (XLV, Book I.) Produce C D to A, so that $CA \times AD$ shall be equal to the square of D E (298). From A inflect $AB = DE$ on the perpendicular D B, and draw B C, the required lines will then be B D and B C.



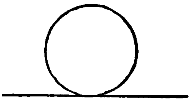
For since the square of A B is equal to $CA \times AD$, the angle A B C is right, ∵ $AB \times DC = BD \times BC$. But $AB = DE$, ∵ the rectangle $CE = BD \times BC$, and C E is equal to the given rectangle. It is evident that the difference of the squares of B D and B C is equal to the square of D C, which is equal to the given difference of squares.

BOOK III.

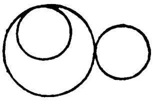
DEFINITIONS.

(309) I. Equal circles are those whose diameters are equal.

(310) II. A right line is said to touch a circle when it meets the circle and, being produced, does not cut it.



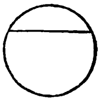
(311) III. Circles are said to touch one another which meet but do not cut one another.



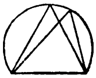
(312) IV. Right lines are said to be equally distant from the centre of a circle when the perpendiculars drawn to them from the centre are equal,

(313) V. And the right line on which the greater perpendicular falls is said to be farther from the centre.

(314) VI. A segment of a circle is the figure contained by a right line and the part of the circumference it cuts off.

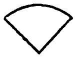


(315) VII. An angle in a segment is the angle contained by two right lines drawn from any point in the circumference of the segment to the extremities of the right line which is the base of the segment.



(316) VIII. An angle is said to stand on the part of the circumference, or the arch, intercepted between the right lines that contain the angle.

(317) IX. A sector of a circle is the figure contained by two radii and the arch between them.



(318) X. Similar segments of circles are those which contain equal angles.

Circles which have the same centre are called *concentric circles.*

(319) The subject of the third book of the elements is the properties of the circle, those of the triangle and rectangle having been discussed in the first and second books respectively.

(320) The first definition is more properly a theorem. For 'equal circles,' like other equal figures, are those which may be laid one upon the other so as perfectly to coincide. If two circles have equal radii, and the centre of one be laid on the centre of the other, the circles being placed in the same plane, their entire circumferences must be coincident; for if not, a line might be drawn from the common centre to the circumference of one, intersecting that of the other, and thus the circles would have unequal radii, contrary to hyp.

(321) In the second definition the meaning of a right line 'cutting a circle' is not explained, and yet it seems as necessary to be defined as 'touching a circle.' If a right line meet the circumference of a circle, and being produced indefinitely in both directions lie entirely without the circle, it is said to *touch* it. The line in this case evidently lies entirely on the *convex* side of the circle.

On the other hand a right line which, when produced, meets a circle in two points, is said to *cut* the circle. The nature of *contact* and *section* will appear more plainly as the student proceeds with the third book.

(322) The same defect is observable in the third definition. Two circles are said to *touch internally* when every point of the one, except those at which they meet, is included within the other; and they touch externally when every point of each, except those at which they meet, lie without the other. It will appear by the thirteenth proposition, that contingent circles can only meet at one point.

(323) Any part of the circumference of a circle is called an *arc* of the circle, and the right line which joins its extremities is called its *chord.* It is evident that two arcs, which together make up the whole circumference, have the same chord.

A diameter is the chord of a semicircle.

(324) The distance of a right line from a point is estimated by the perpendicular from the point on the right line. Chords, therefore, are said to be equally or unequally distant, according as the perpendiculars on them from the centre are equal or unequal.

The figure included by an arc and its chord is a *segment*, and the figure included by an arc and the radii through its extremities is called a *sector.*

(325) It will be proved in Prop. XXI, that all angles inscribed in the same segment of a circle are equal; and also it will appear, that different segments of the same circle contain unequal angles. Thus a segment becomes as it were characterised by the angle it contains, and those segments of different circles which contain the same angles are said to be *similar.* In the sixth book we shall show, that such segments bear the same proportion to the entire circles, of which they are parts.

(326) A quadrant is a sector whose radii form a right angle.

(327) Sectors which have equal radii and equal angles are equal, for they evidently admit of superposition.

(328) A sector whose angle is right, is therefore a fourth part of the circle, and its arc is called a 'quadrant.'

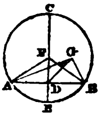
PROPOSITION I. PROBLEM.

(329) To find the centre of a given circle (A C B).

Draw within the circle any right line A B, bisect it in D, from D draw D C perpendicular to A B, and produce it to E; bisect C E in F, and F is the centre.

For, if it be possible, let any other point G be the centre, and draw G A, G D, and G B.

Because in the triangles G D A, G D B, the side G A is equal to G B (hyp. and Def. XV, Book I.), D A equal to D B (const.), and the side G D common to both, the angles G D A and G D B are equal (VIII, Book I.), and therefore are right angles; but the angle C D B is a right angle (const.), therefore G D B is equal to C D B (53), a part equal to the whole, which is absurd; G therefore is not the centre of the circle A C B; and in the same manner it can be proved that no other point which is not on the line C E is the centre, therefore the centre is in the line C E, and therefore is the point F.



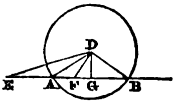
Cor.—From this it is manifest, that if in a circle a straight line bisect another at right angles, the centre of the circle is in the line which bisects the other.

It is assumed in the solution of this problem, that the perpendicular through D will meet the circle at two points. It would have contributed much to the rigour of the reasoning, had Euclid established the following proposition previously to the above problem.

(330) Any point being assumed within a circle, a right line drawn through it, and produced indefinitely in both directions, will meet the circle in two points, and not in more, and every point of the line between these two points of intersection will be within the circle, and every point beyond them without it.

First.—Let the right line through the given point also pass through the centre. If parts be taken upon it in both directions from the centre greater than the radius, their extremities will be *without* the circle (22), and if parts be taken on it in both directions from the centre less than the radius, their extremities will be *within* the circle (22). If parts be taken on it in both directions equal to the radius, their extremities will be *on* the circle. Hence, in this case, the proposition is manifest.

Secondly.—If the line through the given point F within the circle do not pass through the centre D, let a perpendicular D G from the centre to that line be supposed to be drawn. D G is less than D F, and therefore less than the radius. Let a line be found whose square is equal to the difference of the squares of D G and the radius, and take on each side of G, G B and G A equal to this line, and draw D B and D A. Since the squares of B G



and G D are together equal to the square of the radius, the lines D B and D A must be equal each to the radius (XLVII, Book I.), and therefore B and A are *on* the circle. The distance of every point, as F between B and A from D, is less than the radius D A, and therefore (22) every such point is *within* the circle; and the distance of every point, as E in the production of A B from D, is greater than the radius D A, and therefore (22) every such point is *without* the circle. Hence it is plain, that the right line can only meet the circle at the points A and B.

It is no objection to this theorem, that we assume the centre D of the circle without previously solving the problem to find it. In fact, we only assume that the circle *has a centre*, which is given by its definition. It is not necessary to the validity of the demonstration of a theorem, that we should have solutions of all the problems requisite for its construction.

In fact, if all the problems in Geometry were omitted, the reasoning in the theorems would stand undisturbed, and would be equally valid and conclusive.

To the validity of the reasoning contained in the theorems, however, it is indispensably necessary that nothing should be assumed in the construction which is not *possible* to be executed. Thus, if we were required to draw a right line through three given points, we would not be warranted in supposing this done, unless it were also given or proved that the three points have such a position, that the right line through two of them will also pass through the third.

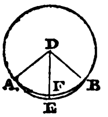
It follows from what has just been proved, that a right line cannot meet a circle in more than two points.

PROPOSITION II. THEOREM.

(331) If any two points (A and B) be taken in the circumference of a circle, the right line which joins them falls within the circle.

For, if it be possible, let A E B be a right line in which the point E is without the circle, and draw D A, D E, and D B.

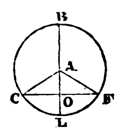
Because in the triangle A D B the sides D A and D B are equal, the angle D B A is equal to D A B (V, Book I.); but the external angle D E A is greater than the internal angle D B A (XVI, Book I.), therefore greater than the angle D A B, and therefore the side D A is greater than the side D E (XIX, Book I.); but the right line D F is equal to D A, and therefore is greater than D E, a part greater than the whole, which is absurd, therefore the line A E B is not a right line; and in the same manner it can be demonstrated, that if the point E be in the circumference the line is not a right line.



This proposition has been already proved by direct reasoning in (330).

PROPOSITION III. THEOREM.

(332) If a right line (B L) drawn through the centre of a circle bisect a right line (C F) which does not pass through the centre, it is perpendicular to it. And if it intersect it at right angles, it bisects it.



Part 1°.—Draw A C and A F. In the triangles A O C, A O F, the side A C is equal to A F, and also O C to O F (hyp.), and A O is common to both, therefore the angle A O C is equal to A O F (VIII, Book I.); therefore each of them is a right angle, and therefore B O is perpendicular to C F.

Part 2°.—Because the triangle F A C is isosceles the angle A F C is equal to the angle A C F (V, Book I.), therefore in the triangles C A O, F A O, the angles A C O and A F O are equal; also A O C and A O F are equal (hyp.), and the side A O, opposite to the equal angles A C O and A F O, is common to both, therefore the side O C is equal to O F (XXVI, Book I.), and therefore the right line C F is bisected.

Hence it appears, that if a system of parallel chords be drawn in a circle, the locus of their points of bisection is the diameter of the circle which is perpendicular to them.

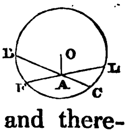
It also follows, that the right line which bisects any chord perpendicularly, bisects every chord parallel to it perpendicularly, and is a diameter of the circle.

PROPOSITION IV. THEOREM.

(333) If in a circle two right lines cut one another, which do not both pass through the centre, they do not bisect one another.

If one of the lines pass through the centre, it is evident that it cannot be bisected by the other, which does not pass through the centre.

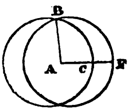
But if neither of the lines B C or F L pass through the centre, draw O A from the centre to their intersection. If B C be bisected in A, O A is perpendicular to it (III), and therefore the angle O A C a right angle; and if F L be bisected in A, O A is perpendicular to F L (III), therefore the angle O A L is a right angle, and therefore equal to the angle O A C, a part equal to the whole, which is absurd, therefore the lines B C and F L do not bisect one another.



Hence it follows, that no parallelogram, except a rectangle, can be inscribed in a circle. For the diagonals bisect each other (155), and therefore must both pass through the centre, and must therefore be equal, each being a diameter. Hence the parallelogram must be a rectangle (157).

PROPOSITION V. THEOREM.

(334) If two circles (A B C, A B F) cut one another, they have not the same centre.



For, if it be possible, let A be the centre of both circles, and draw two right lines, the one A F cutting both circles in C and F, the other A B to the intersection B.

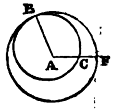
Because A is the centre of the circle A B C, A B is equal to A C, and because A is the centre of the circle A B F, A B is equal to A F, therefore A C is equal to A F, a part to the whole, which is absurd; A therefore is not the centre of both circles; and in the same manner it can be proved that no other point is the centre of both.

This proposition may be better announced thus: 'Concentric circles cannot meet, and that which has the lesser radius will be included within the other.' If the circles had the same radius they would coincide, and, in fact, be the same circle.

The points of the circumference of that which has the lesser radius, being less distant from the centre than those of the circumference of that which has the greater radius, must be all within the latter (22). Consequently, the circles cannot meet, either by contact or intersection. This proof also includes the following proposition.

PROPOSITION VI. THEOREM.

(335) If two circles (A B C, A B F) touch one another internally, they have not the same centre.



For, if possible, let A be the centre of both circles, and draw two right lines, the one A F cutting both circles in C and F, the other A B to the point of contact.

Because A is the centre of the circle A B C, A B is equal to A C, and because A is the centre of the circle A B F, A B is equal to A F, therefore A C is equal to A F, a part equal to the whole, which is absurd; therefore the point A is not the centre of both circles; and in the same manner it can be demonstrated that no other point is.

Vide observations on the last proposition.

PROPOSITION VII. THEOREM.

(336) **If from any point within a circle which is not the centre right lines be drawn to the circumference, the greatest is that which passes through the centre.**

The remaining part of the diameter is the least.

Those lines which make equal angles with the diameter are equal.

That line which is nearer to the line passing through the centre is greater than one more remote.

And more than two right lines cannot be drawn which shall be equal.

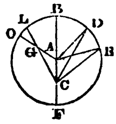
Part 1°.—The line C B passing through the centre is greater than any other C D.

Draw from the centre A the line A D; A B is equal to A D, therefore if C A be added to both, C B shall be equal to C A and A D taken together; but C A and A D are greater than C D (XX, Book I.), therefore C B is greater than C D.

In fact, A D is equal to the sum of the sides of a triangle, of which any other line, as C D, is the base.

Part 2°.—The other part of the diameter C F is less than any other line C E.

Draw A E; A C and C E taken together are greater than A E (XX, Book I.), and therefore greater than A F; take away the common line A C from both, and C E shall be greater than C F



The line C F is the difference of the sides of a triangle, of which any other line C E is the base (99).

Part 3°.—The right lines C L and C D, which make equal angles with the line C B passing through the centre, are equal.

For, if possible, let one of them C L be the greater, and make C G equal to C D, and draw A D and A G.

In the triangles A C G and A C D the side A C is common to both, C G is equal to C D (hyp.), and the angles A C G and A C D are equal, therefore the sides A G and A D are equal (IV, Book I.); but A D is equal to A O, and therefore A G is equal to A O, a part equal to the whole, which is absurd. Therefore neither C L nor C D is greater than the other, and therefore they are equal.

Part 4°.—The line C D or C L, which is nearer to the line passing through the centre, is greater than one C E more remote.

If the given lines C D and C E be at the same side of C B, draw A D and A E. In the triangles C A D, C A E, the sides C A and A D are equal to C A and A E, and the angle C A D is greater than C A E, therefore the side C D is greater than C E (XXIV, Book I.).

But if the given lines C L and C E be at different sides of C B, construct the angle A C D equal to A C L, and C D shall be equal to C L (Part 3°.); but C D is greater than C E, and therefore C L is greater than C E.

Part 5°.—More than two right lines cannot be drawn which shall be equal.

For let any three right lines be drawn from the point C to the circumference, and either one of them shall be part of a diameter, and therefore greater or less than either of the others (by Part 1°. and 2°.), or two of them must be at the same side of the diameter, and therefore unequal (by Part 4°.).

The results of this proposition may be expressed thus:—

If a line always terminated in the circumference revolve round a point C, within a circle different from the centre, it will vary in its magnitude between certain limits. As it revolves from the position C A B towards F in either direction, it diminishes, and at equal distances at each side of C A B it has equal magnitudes; and this diminution continues until, having made half a revolution, it assumes the position C F. In the positions C B and C F it is therefore a maximum and minimum; and the nearer it is to the maximum position the greater it is, and the nearer to the minimum position the less it is.

Proposition VIII. Theorem.

(337) If from any point without a circle lines be drawn to the circumference, those which make equal angles with the line passing through the centre are equal.

Of those lines which are incident upon the concave circumference, the greatest is that which passes through the centre.

Of the rest, that which is nearer to the line passing through the centre is greater than the more remote.

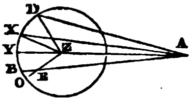
But of those incident upon the convex circumference, that line is the least which, if produced, would pass through the centre.

Of the rest, that which is nearer to the least is less than the more remote.

Only two lines can be drawn either to the concave or convex circumference which shall be equal.

Part 1°.—The right lines A B and A X, which make equal angles with A Z, are equal.

For, if it be possible, let one of them A B be greater than the other; make A E equal to A X, and draw Z E and Z X.



In the triangles Z A E, Z A X, the side Z A is common, A E is equal to A X (const.), and the angle Z A E is equal to Z A X (hyp.), therefore the sides Z E and Z X are equal (IV, Book I.); but the line Z O is equal to Z X, therefore Z E is equal to Z O, a part equal to the whole, which is absurd. Therefore neither A B nor A X is greater than the other, and therefore they are equal.

Part 2°.—Of those lines which are incident upon the concave circumference, that line A Y which passes through the centre is greater than any other A X.

Draw Z X; and Z Y is equal to Z X, therefore if A Z be added to both, A Y shall be equal to A Z and Z X taken together; but A Z and Z X together are greater than A X (XX, Book I.), therefore A Y is greater than A X.

A Y is the sum of the sides of a triangle, of which any other line A X is the base.

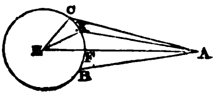
Part 3°.—The line A B or A X, which is nearer to the greatest, is greater than the more remote A D.

If the given lines A X and A D be at the same side of A Y, draw Z X and Z D. In the triangles A Z X, A Z D, the sides A Z, Z X are equal to the sides A Z, Z D, and the angle A Z X is greater than A Z D, therefore the side A X is greater than A D (XXIV, Book I.).

But if the given lines A B and A D be at different sides of A Y, make the angle Z A X equal to Z A B, and A X shall be equal to A B (Part 1°.); but A X is greater than A D, therefore A B is greater than A D.

Part 4°.—Of those lines which are incident on the convex circumference, that line A F, which if produced would pass through the centre, is less than any other A X.

Draw Z F and Z X. Z X and X A are greater than Z A (XX, Book I.), and therefore if the equals Z X and Z F be taken away, A X is greater than A F.



A F is the difference of the sides of a triangle, of which any other line A X is the base.

Part 5°.—That line A B or A X which is nearer to the least is less than the more remote A C.

If the given lines A X and A C be at the same side of A Z, draw Z X and Z C. Z C and C A taken together are greater than Z X and X A; take away the equals Z C and Z X, and A C is greater than A X. But if the given lines A B and A C be at different sides of A Z, make the angle Z A X equal to Z A B, and A X shall be equal to A B (Part 1°.); but A C is greater than A X, and therefore greater than A B.

Part 6°.—Only two equal lines can be drawn either to the concave or convex circumference.

If any three lines be drawn, either one of them shall pass through the centre, and therefore be either greater or less than either of the others, or two must be at the same side of the line passing through the centre, and therefore unequal.

Hence if a line be supposed to revolve round the fixed point A, as it recedes from A Y in either direction it diminishes. When it recedes so far that the part intercepted within the circle vanishes, and the two points of intersection with the circle unite and become one, the line becomes a tangent. If it recede beyond this, it will not meet the circle at all. (The line is called a *secant* so long as it meets the circle in two points). As the line revolving from the tangential position again approaches A F, being terminated in the convex part of the circumference, it still diminishes; and becomes a *minimum* where it assumes the position C F. Thus it appears, that the tangent is less than any secant from the same point, but greater than the external part of the secant.

PROPOSITION IX. THEOREM.

(338) **If a point be taken within a circle, from which more than two equal right lines can be drawn to the circumference, that point is the centre of the circle.**

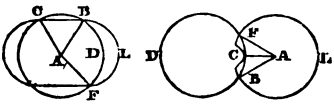
For if it were a point different from the centre, only two equal right lines can be drawn from it to the circumference (VIII).

Thus the criterion for the determination of the centre is, that more than two points of the circumference should be equally distant from it.

PROPOSITION X. THEOREM.

(339) **One circle (B D F) cannot intersect another (B L F) in more than two points.**

For, if it be possible, let it intersect the other in three points, B, F, and C; let A be the centre of the circle B L F, and draw from it to the points of intersection the lines A B, A F, and A C; these lines are equal (Def.), but as the circles intersect, they have not the same centre (V), therefore A is not the centre of the circle B D F, and therefore as three right lines A B, A F, A C are drawn from a point not the centre, these lines are not equal, (VIII); but it was shown before that they were equal, which is absurd; the circles therefore do not intersect in three points.

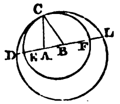


By this proposition two circles cannot intersect in more than two points; but the same demonstration will show that they cannot *touch* in more than two points; hence, in general, two circles cannot have more than two points in common.

Hence also it appears, that if two circles coincide at three points they will coincide at every point, or only one circle can be drawn through three given points. The problem to describe a circle through three given points is the same as to circumscribe a circle round a triangle, and has been solved in (77).

PROPOSITION XI. THEOREM.

(340) If two circles (E C F) and (D C L) touch one another internally, the right line joining their centres, being produced, shall pass through a point of contact.



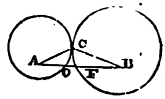
For, if it be possible, let A be the centre of the circle E C F, B the centre of the circle D C L, let D L be the line joining the centres, and from C, a point of contact, draw the lines C B and C A.

Because in the triangle B A C the sides B A and A C, taken together, are greater than B C (XX, Book I.), and B C is equal to B D, as they are radii of the circle D C L, the lines B A and A C, taken together, are greater than B D; take away B A, which is common to both, and A C shall be greater than A D; but A C is equal to A E, because they are the radii of the circle E C F, and therefore A E is greater than A D, a part greater than the whole, which is absurd. The centres are not, therefore, so placed that a line joining them can pass through any point but a point of contact.

In the enunciation and demonstration of this and the next proposition, in Simson's and other translations, the definite article 'the' is applied to the point of contact through which the line joining the centres is proved to pass: thus it is said, that 'the line passing through the centres, being produced, shall pass through *the* point of contact.' In this phraseology there is a silent assumption that there is *but one* point of contact, which is true, but is not established until the thirteenth proposition.

PROPOSITION XII. THEOREM.

(341) If two circles (A O C and B F C) touch one another externally, the right line joining their centres passes through a point of contact.



For, if it be possible, let A and B be the centres, and let the right line A B joining them not pass through a point of contact, and from C, a point of contact, draw C A and C B to the centres.

Because in the triangle A C B the sum of the sides A C and C B is greater than A B (XX, Book I.), and the line A O is equal to A C, as they are radii of the circle A O C, and the line B F is equal to B C, as they are radii of the circle

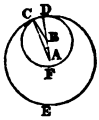
B F C, A O and B F taken together are greater than B A, a part greater than the whole, which is absurd. The centres are not, therefore, so placed, that the line joining them can pass through any point but a point of contact.

From this and the last proposition it follows, that the line joining the centres of contingent circles is the sum of the radii when the contact is external, and the difference of the radii when it is internal.

PROPOSITION XIII. THEOREM.

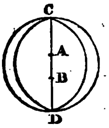
(342) One circle cannot touch another either externally or internally in more points than one.

For, if it be possible, let the circles A D E and B D F touch one another internally in two points D and C; draw the line A B joining their centres, and produce it until it pass through one of the points of contact D, and draw A C and B C.

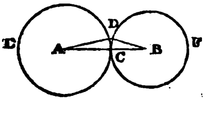


Because B D and B C are radii of the same circle B D F, B D is equal to B C, and therefore, if A B be added to both, A D shall be equal to A B and B C; but A D and A C are radii of the circle A D E, therefore A D is equal to A C, and therefore A B and B C are equal to AC; but they are greater than it (XX, Book I.), which is absurd.

But if the points of contact be the extremities of the right line joining the centres, C D must be bisected in A, and also in B, because it is a diameter of both circles, which is absurd.



Next, if it be possible, let the two circles A D E and B D F touch one another externally in two points D and C; draw the right line A B joining the centres of the circles, and passing through one of the points of contact C, and draw AD and DB.

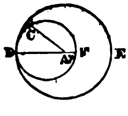


Because A D and A C are radii of the circle A D E they are equal; and because B C and B D are radii of the circle B D F they also are equal, therefore A D and B D together are equal to AB; but they are greater than it (XX, Book I.), which is absurd. There is, therefore, no case in which two circles can touch one another in two points.

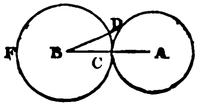
In the 11th and 12th propositions it was proved, that the line joining the centres of contingent circles passed through a point of contact; and in the present we show that this is the only point of contact, by proving that an absurdity would follow from supposing the existence of any other.

This proposition, however, admits of direct proof. We shall express the proposition thus:

If a circle touch another internally, every point of its circumference, except the common point through which the line drawn joining their centres passes, must be within the other.



Let C be any point on the circumference of the lesser circle; A D is greater than A C (VII). Since A C is less than the radius A D of the circle A D E, the point C is within the circle A D E (22); and, in the same manner, every point of the lesser circle may be proved to be within the greater, except the point D, at which they meet.



If the circles touch externally, B D is greater than B C (VIII), ∴ D lies without the circle B C F; and, in like manner, every point of the circle A C D, except the point C, at which they meet, may be proved to lie without B C F.

By the same kind of reasoning it will not be difficult to prove, that if the line joining the centres of two circles be equal to the difference of their radii, they have internal contact, and if it be equal to the sum of their radii, they have external contact.

The following propositions may also be established. If one circle be contained within another without meeting it, the distance between their centres is less than the difference of their radii.

If the distance between the centres be less than the difference between the radii, the lesser circle will be contained within the greater without meeting it.

If two circles lie one without the other, and do not meet, the distance between the centres is greater than the sum of the radii.

If the distance between the centres of two circles be greater than the sum of the radii, they lie one without the other, and do not meet.

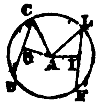
These propositions may be all proved by (VII) and (VIII), united with the criterion established in (22), for determining whether a point be within or without a circle.

PROPOSITION XIV. THEOREM.

(343) In a circle equal right lines (B C and F L) are equally distant from the centre.

And right lines (B C and F L) which are equally distant from the centre are equal.

Let A be the centre of the circle; join A C, A L, and draw A O and A I perpendicular to B C and F L.



Part 1°.—Because B C and F L are equal (hyp.), and the perpendiculars from the centre bisect them (III), O C and I L are equal, and therefore their squares are equal; A C and A L are also equal, and therefore their squares are equal; but the square of A C is equal to the squares of A O and O C (XLVII, Book I.), and the square of A L is equal to the squares of A I and I L (XLVII, Book I.), therefore the squares of A O and O C are equal to the squares of A I and I L: take away the equal squares of O C and I L, and the squares of A O and A I are equal, and therefore the lines themselves are equal.

Part 2°.—Because A O and A I are equal (hyp.), their squares are equal; but A C and A L are equal, and therefore their squares are equal; but the square of A C is equal to the squares of A O and O C (XLVII, Book I.), and the square of A L is equal to the squares of A I and I L, therefore the squares of A O and O C are equal to the squares of A I and I L: take away the equal squares of A O and A I and the squares of O C and I L are equal, therefore the lines themselves are equal; but because A O and A I bisect B C and F L (III), O C and I L are the halves of B C and F L; and since they are equal, the lines B C and F L are also equal.

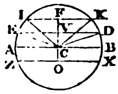
Proposition XV. Theorem.

(344) The diameter is the greatest right line in a circle: and of all others, that which is nearest to the centre is greater than the more remote.

Part 1°.—The diameter A B is greater than any line E D.

For draw C D and C E. C D is equal to C B and C E to C A, therefore A B is equal to C D and C E together; but C D and C E together are greater than E D (XX, Book I.), therefore A B is greater than E D.

Part 2°.—That which is nearer the centre is greater than one more remote.



First, let the given lines be E D and I K, which are at the same side, and do not intersect; draw C D, C E, C I, and C K.

In the triangles E C D, I C K, the sides E C and C D are equal to I C, C K; but the angle E C D is greater than I C K, therefore the side E D is greater than I K (XXIV, Book I.).

Let the given lines be X Z and I K, which either are at different sides, or intersect; draw C O and C F perpendicular to

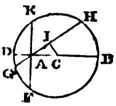
Z X and I K, and from the greater C F cut off C V equal to the less C O, and through V draw E D perpendicular to C F.

Because Z X and E D are equally distant from the centre (const.), E D is equal to Z X (XIV); but E D is greater than I K, and therefore Z X is greater than I K.

This proposition might have been proved in a manner similar to the preceding. The sum of the squares of a semichord and its distance from the centre is equal to the square of the radius. This sum being therefore always the same, whatever the chord be, it follows that the greater the square of the semichord, the less will be the square of its distance, and *vice versa;* and the square of the semichord is greatest when its distance from the centre vanishes. Hence the results of the proposition may easily be inferred.

(345) The shortest chord which can be drawn through a given point A in a circle, is that which is perpendicular to the longest.

The longest is the diameter. Draw the diameter B D, and E F perpendicular to it. Draw any other chord G H, and the perpendicular CI. C A is greater than C I, and (XV) therefore G H is greater than E F; and since the same is true of any other chord, it follows that E F is the least.



The less the angle a chord makes with the diameter through A, the greater the chord will be.

For it is easy to see, that as the angle H A B diminishes the perpendicular C I will also diminish.

PROPOSITION XVI. THEOREM.

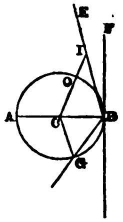
(346) **The right line drawn from the extremity of the diameter of a circle perpendicular to it falls without the circle.**

And if any right line be drawn from a point within that perpendicular to the point of contact, it cuts the circle.

Part 1°.—For, if it be possible, let B G, which meets the circle again, be perpendicular to A B, and draw C G.

Because in the triangle C B G the side G C is equal to C B, the angle C B G is equal to C G B (V, Book I.), and therefore each of them is acute (XVII, Book I.); but C B G is a right angle (hyp.), which is absurd, therefore the right line drawn through B perpendicular to A B does not meet the circle again.

Part 2°.—Let B F be perpendicular to A B, and let E B be a line drawn from a point between it and the circle, which, if it be possible, does not cut the circle.

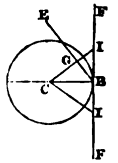


Because the angle C B F is a right angle, C B E is acute; draw C I perpendicular to B E, and it must fall at the side of the angle C B E.

Then in the triangle B C I the angle C I B is greater than C B I, therefore the side C B is greater than C I; but C O is equal to C B, and therefore C O is greater than C I, a part greater than the whole, which is absurd. Therefore the point I does not fall outside the circle, and therefore the right line B E cuts the circle.

(347) This proposition might have been proved directly, thus:

Draw any line C I to the right line B F. Since C B I is a right angle C B is less than C I, ∵ the point I is without the circle; and the same may be proved of every point of the right line except the point B.



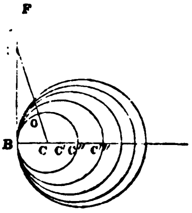
Let B E make the angle E B C acute, and draw the perpendicular C G. Hence C G is less than C B, and therefore G is within the circle, and therefore B E intersects the circle.

The line F F is a tangent to the circle, and it follows, that a tangent can meet the circle only in one point.

Hence, to draw a tangent to a point on a circle, it is only necessary to draw a diameter through that point and to draw a line perpendicular to it.

(348) From this proposition a method has been derived of proving the infinite divisibility of linear magnitude.

Let B F be a tangent at B to the circle whose centre is C. Draw any line CI meeting the circle at O. The line O I may be infinitely divided by describing circles with centres at C', C'', C''', &c., touching B F at B. This is obvious.

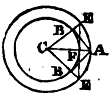


If several circles touch each other, either internally or externally, they have at their point of contact a common tangent. For the same right line is perpendicular to that which passes through their centres.

(349) From this proposition it follows, that tangents through the extremities of the same diameter are parallel.

Proposition XVII. Problem.

350) From a given point (A), without a given circle (C B F), to draw a right line which shall be a tangent to the circle.



Let C be the centre of the given circle, and from the centre C with the radius C A describe a circle C A E; draw C A, which meets the circle in the point F, and draw through the point F the line F E perpendicular to C A, and meeting the circle C A E in E; draw the line C E meeting the given circle in B, and the right line drawn from B to the given point A is a tangent.

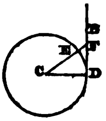
For in the triangles A C B, E C F the sides A C and C B are equal to E C and C F, and the angle at C is common to both; therefore the angle A B C is equal to E F C (IV, Book I.); but the angle E F C is a right angle (const.), therefore A B C is a right angle, and therefore the right line A B is a tangent to the circle C F B (XVI).

(351) It is evident that two tangents and not more can be drawn from the point A; for the perpendicular to C A through F meets the circle C A E in two points and no more, and each of these points will determine a tangent.

The two triangles A B C are evidently equal in every respect. Hence the two tangents A B are equal, and equally inclined to the line A C through the centre, lying on different sides of it.

Proposition XVIII. Theorem.

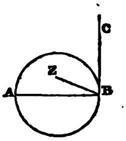
(352) If a right line (D B) be a tangent to a circle, the right line (C D) drawn from the centre to the point of contact is perpendicular to it.



For, if it be possible, let the right line C F be perpendicular to B D, and in the triangle C F D, because the angle C F D is a right angle, the angle C D F is acute (XVII, Book I.), therefore the side C D is greater than the side C F (XIX, Book I.); but C E is equal to C D, and therefore C E is greater than C F, a part greater than the whole, which is absurd. Therefore C F is not perpendicular to B D; and in the same manner it can be demonstrated, that no other line except C D is perpendicular to it.

Proposition XIX. Theorem.

(353) If a right line (B C) be a tangent to a circle, the right line (B A) drawn perpendicular to it from the point of contact passes through the centre of the circle.



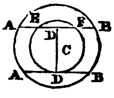
For, if it be possible, let the centre Z be without the line B A, and draw Z B.

Because the right line Z B is drawn from the centre to the point of contact, it is perpendicular to the tangent (XVIII), therefore the angle Z B C is a right angle; but the angle A B C is also a right angle (hyp), and therefore Z B C is equal to A B C, a part to the whole, which is absurd.

Therefore Z is not the centre; and in the same manner it can be demonstrated, that no other point without the line A B is the centre.

(354) From this and the preceding propositions we may deduce the following consequences:—

Two concentrical circles being described, if a chord of the greater meet the less, the parts intercepted between the two circles are equal.



Let C D be perpendicular to A B.

1°. Let A B intersect the lesser circle. Then A D = B D and E D = F D (III), ∴ A E = F B.

2°. Let A B touch the lesser circle. The angle C D A is right (XVI), ∴ A D = D B.

Hence all chords of the greater circle which touch the lesser, are bisected at the points of contact.

All such chords are equal, since their distances from the centre are equal to the radius of the lesser circle (XIV).

⁂ It is obvious that if any number of equal chords be drawn in a circle the *locus* of their points of bisection is a circle, the square of whose radius is equal to the difference between the squares of the radius of the given circle and half the chord.

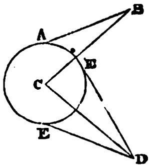
⁂ *Through a given point within or without a circle to draw a chord of a given length.*

In order that the solution of this problem be possible, it is necessary that the given length should not be greater than the diameter of the circle (XV); and if the given point be within the circle, it is further necessary, that it should not be less than the chord through the given point at right angles to the diameter through the same point (344).

When the solution then is possible, let any chord be drawn in the circle equal to the given magnitude, by taking any point on the circle as centre and the given length as radius, and describing another circle, and drawing from the assumed point to a point of intersection of the two circles a chord. From the centre let a perpendicular be drawn to this chord, and with that perpendicular as radius describe a concentrical circle. Through the given point draw a tangent to this circle, and it will be the line required,

The demonstration will easily be inferred from the preceding articles.

⁂ Between a circle and a right line, or between two circles not concentrical, to inflect a line of a given length which shall touch one of the circles. Draw any tangent A B to one of the given circles, and take A B equal to the given length, and draw C B from the centre C. Describe the concentric circle C B D, and draw the tangent D E from

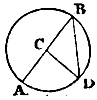


the point D where this circle meets the given right line or the other circle. The demonstration is evident from (351).

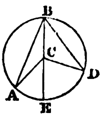
If the circle C B D do not meet the given right line or the other circle the solution is impossible.

Proposition XX.

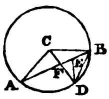
(355) The angle (A C D) at the centre of a circle, is double the angle (A B D) at the circumference, when they have the same part of the circumference for their base.



1°. Let the centre be on A B a side of the angle A B D at the circumference. Because in the triangle D C B the sides D C and C B are equal, the angles C B D and C D B are equal (V, Book I.). But the external angle A C D is equal to the sum of C B D and C D B, or to twice C B D (XXXII, Book I.).



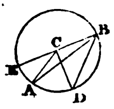
2°. Let the centre be within the angle A B D; draw B C E; the angles A B C and C A B are equal, and the angles C B D and C D B are also equal, because of the equality of the sides C D, C B, C A, (V, Book I). Hence the sum of the angles C A B, C B A, C B D, and C D B is double the angle A B D. But E C A is equal to the sum of C B A and C A B, and E C D is equal to the sum of C B D and C D B, therefore A C D is equal to the sum of C A B, C B A, C B D, and C D B, and therefore A C D is double of A B D.



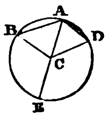
3°. Let the centre be without the angle A B D. Draw D E and C B, making the angle C D E equal to C B A, and join D B. D F E is equal to C F A (XV, Book I.). But also since C A is equal to C B, the angle C A B is equal to C B A (V, Book I.), which is (const.) equal to C D E, therefore C A F is equal to F D E. Hence the angle F E D is equal to the angle A C D. But the angle C D B is equal to C B D, since the sides C D and C B are equal: taking from these equals the angles C D E and C B E, the remainders A B D and E D B are equal. The angle A E D is equal to A B D and E D B taken together, and therefore to twice the angle A B D. But A E D is equal to A C D, which is therefore equal to double the angle A B D.

(356) We have taken this demonstration from Clavius, that given by Euclid being considered objectionable, in consequence of assuming some relations not warranted by the axioms.

Euclid reasons thus:—When the centre falls within the angle at the circumference, A C E is double A B C, and D C E is double C B D; by adding these we have A C D double A B D.



If the centre fall outside the angle at the circumference, we have E C A double E B A, and E C D double E B D; taking the former from the latter we have A C D double A B D. This is more concise, but not so rigorous as the reasoning of Clavius.

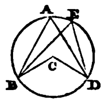


The relation established in this proposition between the central and circumferential angles on the same arc extends to the cases in which the central angle is greater than two right angles (14, 135). Let B C D presented towards E be such an angle. Draw A C E. The reverse angle B C D is equal to the sum of the angles B C E and D C E. But B C E is twice B A C, and D C E is twice D A C, ∵ the reverse angle B C D is equal to twice the angle B A D.

This generalization would give considerable brevity to some of the succeeding demonstrations.

PROPOSITION XXI. THEOREM.

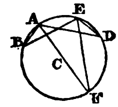
(357) The angles (B A D, B E D) in the same segment of a circle are equal.



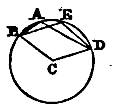
1°. Let the segment B A D be greater than a semicircle, let C be the centre of the circle, and draw C B and C D.

The angle B C D at the centre is double of the angle B A D, and also double of B E D (XX); therefore B A D and B E D are equal to one another.

2°. Let the segment B A D be a semicircle, or less than a semicircle, let C be the centre of the circle, and draw the right lines A C F and E F.

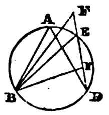


Because the segment B D F is greater than a semicircle, and in it are the angles B A F and B E F, B A F is equal to B E F (Part 1°.); and because the segment F B A D is greater than a semicircle, and in it are the angles F A D and F E D, F A D is equal to F E D (Part 1°.); therefore the sum of the angles B A F and F A D, or the angle B A D, is equal to the sum of B E F and F E D, or to the angle B E D.



(358) If the term 'angle' had been extended by Euclid, as it has been in modern science, to angles greater than two right angles, no subdivision of this demonstration into cases would be necessary. The second case would be proved as the first. Each of the angles B A D and B E D would be equal to half the reverse central angle B C D.

If two equal angles stand on the same arc, and the vertex of one be in the opposite segment, the vertex of the other will also be in it.



For if not at E let it be within or without it at F, and draw B E. The angles B A D and B F D are equal (hyp.), but B A D and B E D are equal (XXI) also. Hence the angles B E D and B F D are equal; but one is greater than the other by (XVI, Book I.), hence, &c.

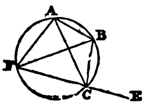
Hence it appears, that if innumerable triangles be constructed on the same base with equal vertical angles, the vertices would form the segment of a circle. In other words, if the base and vertical angle of a triangle be given, the *locus* of the vertex is the segment of a circle.

There is no difficulty in proving that any angle within a segment contained by lines drawn to the extremities of its base, is greater than an angle in it, and *vice versa*, and any such angle without it is less than an angle in it, and *vice versa*.

PROPOSITION XXII. THEOREM.

(359) The opposite angles of a quadrilateral figure (F A B C) inscribed in a circle, are together equal to two right angles.

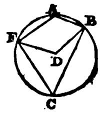
Draw the diagonals A C and F B.



Because the angles A C B and A F B are in the same segment A F C B, A C B is equal to A F B (XXI), and because the angles A C F and A B F are in the same segment A B C F, A C F is equal to A B F (XXI), therefore the angle B C F is equal to the angles A F B and A B F taken together; but the angles A F B and A B F together with F A B are equal to two right angles (XXXII, Book I.), and therefore B C F together with F A B is equal to two right angles: in the same manner it can be demonstrated, that A B C and A F C are equal to two right angles.

(360) If any side F C be produced, the external angle B C E will be equal to the opposite internal angle F A B, for they have a common supplement F C B.

(361) This proposition might be derived from the twentieth, thus:—

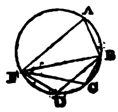


Draw D F and D B. The angle F C B is half of F D B, and the angle F A B is half of the reverse angle F D B, ∵ the angles A and C together are equal to half the sum of the angles round the point D, that is, to two right angles.

(362) If two chords cut off similar segments from the same or different circles, the other segments will also be similar, since the angles they contain are supplemental to those in the former segments.

(363) If opposite angles of a quadrilateral be equal, they must be both right, right angles being the only equal angles which are supplemental.

(364) If the opposite angles of a quadrilateral be supplemental, a circle may be circumscribed about it.

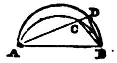


For if a circle be described passing through the vertices of three of its angles A, B, F, it must also pass through the fourth C. Take any point D in the segment F D B, and draw D F, D B. The angle D is supplemental to A (XXII), and ∵ equal to C, and since they are on the same base F B, and D is in the circle, C must also be in it (358).

PROPOSITION XXIII. THEOREM.

(365) Upon the same right line, and upon the same side of it, two similar segments of circles cannot be constructed which do not coincide.

For, if it be possible, let two similar segments A C B and A D B be constructed, and let the point D in one of them fall without the other, and draw the right lines D A, D B, and C B.

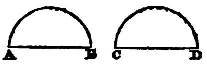


Because the segments A C B and A D B are similar the angle A C B is equal to A D B (Def.); but A C B is external to A D B, and therefore greater than it (XVI, Book I.), which is absurd: therefore no point in either of the segments falls without the other, and therefore the segments coincide.

PROPOSITION XXIV. THEOREM.

(366) Similar segments of circles standing upon equal right lines (A B and C D) are equal.

For if the equal right lines A B and C D be applied one on the other so that the point A may fall on C, the point B must fall upon D, and therefore the right lines coincide; therefore the segments themselves coincide (XXIII), and therefore they are equal.



⁂ (367) Hence it follows, also, that similar segments having equal chords have also equal arcs.

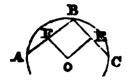
Also, since two circles must coincide in every part, if they agree in more than two points, it follows that similar segments having equal chords are parts of equal circles.

⁂ (368) Sectors whose radii and angles are equal, are themselves equal. For if the chords of the arcs be drawn, they will be divided into triangles and segments. The triangles will be equal (IV, Book I.); and since the angles at the centres are equal, those at the circumference are also equal (XX), and also those in the arcs of the sectors (XXII). Hence the segments are similar, and being on equal right lines, are equal.

PROPOSITION XXV. PROBLEM.

(369) A segment (A B C) of a circle being given, to describe the circle of which it is the segment.

From any point B draw two right lines B A and B C, bisect them, and from the points of bisection F and E draw two lines F O and E O perpendicular to A B and B C; the intersection O of these perpendiculars is the centre.



Because the right line A B terminated in the circle is bisected by a perpendicular F O to it, F O passes through the centre (I), likewise E O passes through the centre (I), therefore the centre must be O, the intersection of these lines F O and E O.

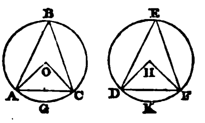
It is here assumed that the perpendiculars E O and F O will meet. This appears from considering that if the line F E be drawn, they will make angles with it, which are together less than two right angles.

Proposition XXVI. Theorem.

(370) In equal circles (A B C, D E F), equal angles A O C and D H F, A B C and D E F), whether they be at the centres or at the circumferences, stand upon equal arches.

First, let the given angles A O C and D H F be at the centres; draw to any points B and E in the circumferences the lines A B and C B and DE, F E, and join A C and D F.

Because in the triangles A O C, D H F the angles O and H are equal (hyp.), and the sides A O and O C equal to D H and H F (hyp.), the bases A C and D F are equal (IV, Book I.); but the angles A B C and D E F are equal (XX), and therefore the segments A B C and D E F are similar (Def.), but they stand upon equal right lines A C and D F, and are therefore equal (XXIV); take away these equals from the equal circles, and the remaining segments are equal, and therefore the arches A G C and D K F are equal.



In the same manner it can be demonstrated, that the arcs A G C and D K F are equal, if the given angles at the circumferences A B C and D E F are acute, by drawing O A and O C and also H D, H F.

But if the given angles at the circumferences are either right or obtuse, bisect them, and the halves of them are equal, and it can be proved as above, that the arcs upon which these halves stand are equal, whence it follows that the arcs on which the given angles stand are equal.

(371) It is evident that this proposition extends to equal central or circumferential angles in the *same* circle, and also to the cases of reverse angles.

⁂ (372) Hence if the opposite angles of a quadrilateral in a circle be equal, the diagonal opposite them must be a diameter; and since in this case the angles are both right, it follows that a segment containing a right angle is a semicircle.

⁂ (373) There is no difficulty in deducing from this proposition, that if one central or circumferential angle in the same or equal circles be greater than another, the arc on which the one stands will be greater than that on which the other stands.

Hence it appears, that if of two opposite angles of a quadrilateral inscribed in a circle, one be acute, and therefore the other obtuse, the arc on which the former stands will be less, and the latter greater, than a semicircle. Hence the segment which contains an acute angle is greater, and that which contains an obtuse angle less, than a semicircle. These results are the converse of Prop. XXXI.

⁂ (374) Supplemental circumferential angles in the same or equal circles, stand on arcs whose sum is equal to a whole circumference.

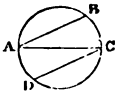
⁂ (375) Diameters intersecting at right angles divide the circumference into four equal arcs.

⁂ (376) Any number of central angles in the same or equal circles, whose sum is equal to four right angles, stand on arcs whose sum is equal to a whole circumference.

⁂ (377) Any number of circumferential angles in the same or equal circles, whose sum is equal to two right angles, stand on arcs whose sum is equal to a whole circumference.

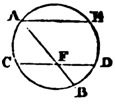
⁂ (378) *Similar arcs of equal circles are equal.*

⁂ (379) *Parallel chords* A B, C D *of a circle intercept equal arcs; and vice versâ.*



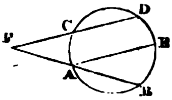
Draw A C. The alternate angles B A C and A C D are equal, therefore the arcs B C and A D on which they stand are equal. Again, if A D = B C, then (XXVII) A C D = B A C; ∵ A B is parallel to D C.

⁂ (380) *If two chords* A B, C D, *intersect within a circle, the sum of the arcs they intercept is equal to the arc which a circumferential angle equal to that under the chords would intercept.*



Draw A E parallel to C D. Then A C = D E, ∵ E B = the sum of A C and B D. But B A E = B F D.

⁂ (381) *If two chords intersect at a point without a circle, the difference of the arcs which they intercept is equal to the arc which a circumferential angle equal to that under the chords would intercept.*



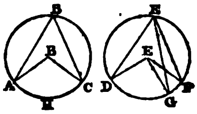
Draw A E parallel to C D. Then A C = D E, ∵ B E is the difference between B D and A C: but B A E = B F D.

⁂ (382) If chords within a circle intersect at the same angle, the sums of the arcs they respectively intercept are equal; and if they intersect without the circle the differences are equal; and if one pair intersect within and the other without, the sum of the one pair of arcs is equal to the difference of the other.

⁂ (383) If chords intersect at a right angle, the sum of the arcs they intercept is equal to a semicircle.

PROPOSITION XXVII. THEOREM.

(384) In equal circles (A B C, D E F), the angles (A B C and D E F) which stand upon equal arches are equal, whether they be at the centres or at the circumferences.



For, if it be possible, let one of them D E F be greater than the other, and make the angle D E G equal to A B C.

Because in the equal circles A B C and D E F the angle A B C is equal to D E G (hyp.), the arcs A H C and D G are equal (XXVI), but A H C and D G F are also equal (hyp.), and therefore D G is equal to D F, a part equal to the whole, which is absurd: neither angle therefore is greater than the other, and therefore they are equal.

⁂ (385) This proposition extends, like the former, to arcs in the same circle, as well as in equal circles, and inferences follow which are converses to those made from (XXVI).—*Ex. gr.:* The sum of the central angles subtended by arcs, whose sum is equal to an entire circumference, is equal to four right angles. The sum of the circumferential angles subtended by the same arcs is equal to two right angles, &c.

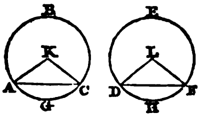
A quadrant subtends a right angle at the centre, and a semicircle at the circumference.

It follows also from this proposition that equal arcs of equal circles contain similar segments.

PROPOSITION XXVIII. THEOREM.

(386) In equal circles (A B C, D E F), equal right lines (A C and D F) cut off equal arcs, the greater equal to the greater (A B C to D E F), the less to the less (A G C to D H F).

If the equal right lines be diameters the proposition is evident.



If not, let K and L be the centres of the circles, and draw the lines K A, K C, L D, and L F.

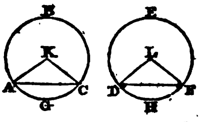
Because the circles are equal (Def.), A K and K C are equal to L D and L F, and also A C and D F are equal (hyp.), therefore the angle A K C is equal to the angle D L F (VIII, Book I.), and therefore the arc A G C is equal to the arc D H F (XXVI); and since the circles are equal, take away these equal arcs from them, and the remaining arcs A B C and D E F are equal.

PROPOSITION XXIX. THEOREM.

(387) In equal circles (A B C, D E F), the right lines (A C and D F) which subtend equal arcs are equal.

If the equal arches be semicircles the proposition is evident.

But if not, let K and L be the centres of the circles, and draw A K, K C, D L, and L F.



Because the arcs A G C and D H F are equal (hyp.), the angles A K C and D L F are equal (XXVII); but in the triangles A K C and D L F the sides A K and K C are equal to D L and L F (hyp.), and therefore the bases A C and D F are equal (IV, Book I.).

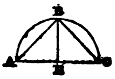
PROPOSITION XXX. PROBLEM.

(388) To bisect a given arc (A B C).

Draw the right line A C; bisect it in E, through E draw E B perpendicular to A C, and it bisects the arc in B.

Draw the right lines A B and C B.

In the triangles A E B, C E B, the sides A E and E C are equal (const.), E B is common, and the angle A E B is equal to C E B (const.), therefore the sides A B and B C are equal (IV, Book I.), and therefore the arcs which they subtend are equal (XXVIII), and therefore the given arc is bisected in B.

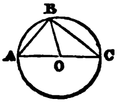


PROPOSITION XXXI. THEOREM.

(389) In a circle the angle in a semicircle is a right angle, the angle in a segment greater than a semicircle is acute, and the angle in a segment less than a semicircle is obtuse.

Part 1°.—The angle A B C in a semicircle is a right angle.

Let O be the centre of the circle, and draw O B and A C.



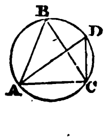
Because in the triangle A O B the sides O B and O A are equal, the angles O A B and O B A are also equal (V, Book I.); in the same manner it can be proved, that the angles O C B and O B C are equal, therefore the angle

A B C is equal the sum of the angles B C A and B A C, and therefore the angle A B C is a right angle (XXXII, Book I.).

Part 2°.—The angle A B C in a segment greater than a semicircle is acute.

Draw A D, a diameter of the circle, and also the lines C D, CA.

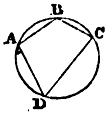
Because in the triangle A C D the angle A C D in a semicircle is a right angle (Part 1°.), the angle A D C is acute (XXXII, Book I.); but the angles A D C and A B C are in the same segment A B D C, and therefore equal (XXI), therefore the angle A B C is acute.



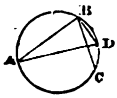
Part 3°.—The angle A B C in a segment less than a semicircle is obtuse.

Take in the opposite circumference any point D, and draw D A and D C.

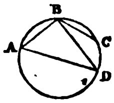
Because in the quadrilateral figure A B C D the opposite angles B and D are equal to two right angles (XXII), but the angle D is less than a right angle (Part 2°.), the angle A B C must be obtuse.



(390) The second part of this proposition might have been more elegantly and concisely proved by drawing D B instead of D C; we should then have A B D a right angle, ∴ A B C acute.



A similar method might be applied to the third case; draw the diameter A D, and draw B D. The angle A B D is right, and ∴ A B C obtuse.



But the proof might be still more elegantly derived from the established relation between central and circumferential (XX) angles having the same subtense. The central angle which stands on a semicircle is equal to two right angles, and therefore the circumferential angle is one right angle. The central angle, which stands on an arc less than a semicircle, is less than two right angles, and therefore the circumferential angle is less than a right angle; and the central angle, which stands on an arc greater than a semicircle, is greater than two right angles; and therefore the circumferential angle is greater than a right angle.

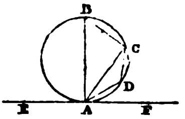
PROPOSITION XXXII. THEOREM.

(391) If a right line (E F) be a tangent to a circle, and from the point of contact a right line (A C) be drawn cutting the circle, the angle F A C made by this line with the tangent is equal to the angle (A B C) in the alternate segment of the circle.

If the chord should pass through the centre, it is evident the angles are equal, for each of them is a right angle (XVI, XXXI.).

But if not, draw through the point of contact the line A B perpendicular to the tangent E F, and join B C.

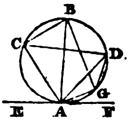
Because the right line E F is a tangent to the circle, and A B is drawn through the point of contact perpendicular to it, A B passes through the centre (XIX), and therefore the angle A C B is a right angle (XXXI); therefore in the triangle A B C, the sum of the angles A B C and B A C is equal to a right angle (XXXII, Book I.), and therefore equal to the angle B A F; take away the common angle B A C, and the remaining angle C A F is equal to the angle A B C in the alternate segment.



The angles E A C and A D C are also equal.

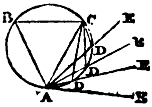
Draw the right lines A D and D C; because in the quadrilateral A B C D the opposite angles A B C and A D C taken together are equal to two right angles (XXII), the sum of the angles E A C and F A C is equal to the sum of A B C and A D C; take away the equals F A C and A B C (Part 1°.), and the remaining angle E A C is equal to the angle A D C in the alternate segment.

(392) This proposition might be otherwise proved, thus:—The angle A C D = D A F. Draw A B perpendicular to E F. The angle A C B = B A F, since both are right; B C D = B A D, being in the same segment. Taking the latter from the former, D A F = A C D.



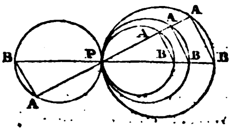
Also draw G B. We have A G B = B A E, both being right; and D G B = D A B, being in the same segment; adding these equals, we have D G A = D A E.

If we consider a triangle inscribed in a circle as a quadrilateral, one of whose sides vanishes, this proposition may at once be derived from the twenty-second. By that proposition in the quadrilateral A B C D, the external angle C D E is equal to the internal opposite angle B. While the point D approaches A, and the side A D diminishes, the angle C D E remains of the same magnitude, and still equal to B. When D coincides with A, and A D vanishes, A E becomes a tangent, and the angle C A E under the chord and tangent is equal to the angle B in the alternate segment.



*** (393) If several circles touch each other, either internally or externally, any right line passing through the point of contact will cut off similar segments from them. For since they have a common tangent (348), the angles in all the segments are equal to the angle under the line drawn and the common tangent.

*** (394) If several circles touch each other internally or externally, and any two right lines be drawn through the point of contact P, cutting each of them at A and B, the lines A B will be parallel; for by (393) the alternate angles P A B are equal.



⁂ (395) Tangents through the extremities of the same chord make equal angles with it on the same side. For each angle is equal to the angle in the alternate segment.

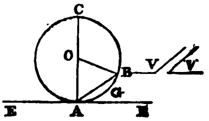
⁂ (396) The chord joining the points of contact of parallel tangents is a diameter. For the angles on the same side are equal (395), and supplemental (XXIX, Book I.), and are therefore right. Therefore the chord is a diameter (XIX).

Proposition XXXIII. Problem.

(397) On a given right line (A B) to describe a segment of a circle that shall contain an angle equal to a given angle (V).

If the given angle be a right angle, bisect the given line; describe a semicircle on it. This will evidently be the segment required, since it contains a right angle (XXXI).

If the given angle V be acute or obtuse, make with the given line A B at either extremity of it A an angle B A E equal to V; through A draw A C perpendicular to E A, and at B make the angle A B O equal to B A C. The circle described from the centre O with the radius O A passes through B, because O A and O B are equal, and its segment A C B contains an angle equal to the given acute angle V, and its segment A G B contains an angle equal to the given obtuse angle V.



Because E A is a tangent to the circle at A (XVI), and from given it is drawn A B cutting the circle, the angle in the segment is equal to the acute angle E A B (XXXII), and therefore to the given acute angle V (const.); and also the angle in the segment A G B is equal to the obtuse angle E A B (XXXII), and therefore to the given obtuse angle V (const.).

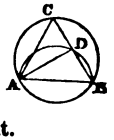
(398) This problem might be solved by constructing on the given right line A B any angle equal to the given angle V, and circumscribing round this triangle a circle (78). Euclid, however, does not introduce the problem to circumscribe a circle round a triangle, until the fourth book.

⁂ (399) From this problem we derive the solution of another—'Given the base and vertical angle of a triangle to construct the *locus* of the vertex.'

This problem is useful, therefore, in the solution of all problems relating to the determination of a triangle, where two of the three data are a side and the angle opposite to it. In such cases, having constructed on the given side the segment which contains the opposite angle, all that remains to be determined is the point in this segment where the vertex is placed. The third datum ought to be sufficient to determine this. Thus, for example, if the third datum be the perpendicular from the vertex on the given side, the place of the vertex may be determined by drawing any line perpendicular to the given side, and taking a part on it from the side equal to the given perpendicular. A parallel to the side through the extremity of this will intersect the circle in two points, either of which will serve for the vertex.

⁂ (400) Again, suppose the base, vertical angle, and the perpendicular from the extremity of the base on the opposite side, be given to find the triangle.

On the given side A B describe a segment A C B containing the given angle. Also describe a semicircle A D B. It is evident that the vertex of the triangle must be in the former, and the point where the perpendicular meets the side in the latter. Inflect A D equal to the perpendicular, and draw B D to meet the first segment at D: the triangle A B C is that required. The demonstration is evident.

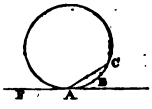


It would be impossible to enumerate the cases in which this principle is useful; and the student cannot obtain a better exercise than in combining with the base and vertical angle of a triangle the various data which may be sufficient to determine the place of the vertex in the segment.

Proposition XXXIV. Problem.

(401) To cut off from a given circle (A B C) a segment which shall contain an angle equal to a given angle.

Draw F A, a tangent to the circle at any point A; at the point of contact make with the line A F an angle F A C, equal to the given angle; the segment A B C contains an angle equal to the given angle.



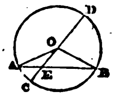
Because F A is a tangent to the circle, and A C cuts it, the angle in the segment A B C is equal to F A C (XXXII), and therefore equal to the given angle (const.).

Proposition XXXV. Theorem.

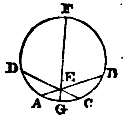
(402) If two right lines (A B and C D) within a circle cut one another, the rectangle under the segments (A E and E B) of one of them is equal to the rectangle under the segments (C E and E D) of the other.

1°. If the given right lines pass through the centre they are bisected in the point of intersection; therefore the rectangles under their segments are the squares of their halves, and therefore are equal.

2°. Let one of the given lines (D C) pass through the centre, and the other (A B) not; draw O A and O B. The rectangle A E B is equal to the difference between the squares of O E and of O A (253), that is, to the difference between the squares of O E and of O C, or to the rectangle D E C (V, Book II.).



3°. Let neither of the given lines pass through the centre, draw through their intersection a diameter F G, and the rectangle under F E and E G is equal to the rectangle under D E and E C, and also to the rectangle under B E and E A (Part 2°.); therefore the rectangle under D E and E C is equal to the rectangle under B E and E A.

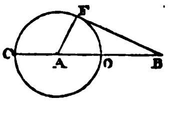


The demonstration of the second case has been somewhat abridged by the principle established in (253). In Euclid's demonstration the proof of that principle is really incorporated.

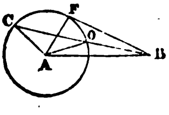
PROPOSITION XXXVI. THEOREM.

(403) If from a point (B) without a circle two right lines be drawn to it, one of which (B F) is a tangent to the circle, and the other (B C) cuts it; the rectangle under the whole secant (B C) and the external segment (B O) is equal to the square of the tangent (B F).

1°. Let B C pass through the centre; draw A F from the centre to the point of contact; the square of B F is equal to the difference between the squares of B A and of AF (XIX), that is, to the difference between the squares of B A and of A O, or to the rectangle under C B and B O (VI, Book II.).



2°. If B C do not pass through the centre, draw A O and A C. The rectangle under C B and B O is equal to the difference between the squares of A B and of A O (253), that is, to the difference between the squares of A B and A F, or to the square of B F (XXI).



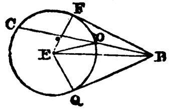
(404) COR.—Hence, if from any point without a circle two right lines be drawn cutting the circle, the rectangles under them and their external segments are equal, for each of the rectangles is equal to the square of the tangent.

The demonstration of the second case has been abridged as in the last proposition.

PROPOSITION XXXVII. THEOREM.

(405) If from a point (B) without a circle two right lines be drawn, one (B C) cutting the circle, the other (B F) meeting it, and if the rectangle under the secant and its external segment be equal to the square of the line which meets the circle, the line (B F) which meets it is a tangent.

Draw from the point B the line B Q, a tangent to the circle, and draw E F and E Q.



The square of B Q is equal to the rectangle under B C and B O (XXXVI), but the square of B F is also equal to the rectangle under B C and B O (hyp.), therefore the squares of B F and B Q are equal, and therefore the lines themselves are equal; then, in the triangles E F B and E Q B the sides E F and F B are equal to the sides E Q and Q B, and the side E B is common, therefore the angle E F B is equal to E Q B (VIII, Book I.); but E Q B is a right angle (XIX), therefore E F B is a right angle, and therefore the right line B F is a tangent to the circle (XVI).

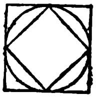
BOOK IV.

DEFINITIONS.

(406) I. A rectilinear figure is said to be inscribed in another, when all the angular points of the inscribed figure are on the sides of the figure in which it is said to be inscribed.

(407) II. A figure is said to be described about another figure, when all the sides of the circumscribed figure pass through the angular points of the other figure.

(408) III. A rectilinear figure is said to be inscribed in a circle, when the vertex of each angle of the figure is in the circumference of the circle.



(409) IV. A rectilinear figure is said to be circumscribed about a circle, when each of its sides is a tangent to the circle.

(410) V. A circle is said to be inscribed in a rectilinear figure, when each side of the figure is a tangent to the circle.

(411) VI. A circle is said to be circumscribed about a rectilinear figure, when the circumference passes through the vertex of each angle of the figure.

(412) VII. A right line is said to be inscribed in a circle, when its extremities are in the circumference of the circle.

(413) The fourth book of the Elements is devoted to the solution of problems, chiefly relating to the inscription and circumscription of regular polygons and circles.

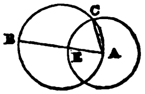
A regular polygon is one whose angles and sides are equal.

PROPOSITION I. PROBLEM.

(414) In a given circle (B C A) to inscribe a right line equal to a given right line, which is not greater than the diameter of the circle.

Draw a diameter A B of the circle, and if this is equal to the given line, the problem is solved.

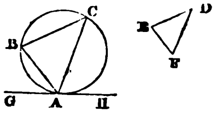
If not, take in it the segment A E equal to the given line (III, Book I.); from the centre A with the radius A E describe a circle E C, and draw to either intersection of it with the given circle the line A C; this line is equal to A E, and therefore to the given line.



PROPOSITION II. PROBLEM.

(415) In a given circle (B A C) to inscribe a triangle equiangular to a given triangle (E D F.)

Draw the line G H a tangent to the given circle in any point A; at the point A with the line A H make the angle H A C equal to the angle E, and at the same point with the line A G make the angle G A B equal to the angle D, and draw B C.

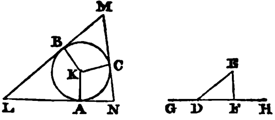


Because the angle E is equal to H A C (const.), and H A C is equal to the angle B in the alternate segment (XXXII, Book III.), the angles E and B are equal; also the angles D and C are equal, therefore the remaining angle F is equal to B A C (XXXII, Book I.), and therefore the triangle B A C inscribed in the given circle is equiangular to the given triangle E D F.

PROPOSITION III. PROBLEM.

(416) About a given circle (A B C) to circumscribe a triangle equiangular to a given triangle (E D F).

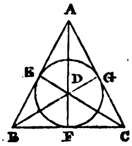
Produce any side D F of the given triangle both ways to G and H; from the centre K of the given circle draw any radius K A. With this line at the point K make the angle B K A equal to the angle E D G, and at the other side of K A make the angle A K C equal to E F H, and draw the lines L M, L N, and M N, tangents to the circle in the points B, A and C.



Because the four angles of the quadrilateral figure L B K A taken together are equal to four right angles (134), and the angles K B L and K A L are right angles (const.), the remaining angles A K B and A L B are together equal to two right angles; but the angles E D G and E D F are together equal to two right angles (XIII, Book I.), therefore the angles A K B and A L B are together equal to E D G and E D F; but A K B and E D G are equal (const.), and therefore A L B and E D F are equal. In the same manner it can be demonstrated that the angles A N C and E F D are equal; therefore the remaining angle M is equal to the angle E (XXXII, Book I.), and therefore the triangle L M N circumscribed about the given circle is equiangular to the given triangle.

PROPOSITION IV. PROBLEM.

(417) In a given triangle (B A C) to inscribe a circle.



Bisect any two angles B and C by the right lines B D and C D, and from their point of concourse D draw D F perpendicular to any side B C; the circle described from the centre D with the radius D F is inscribed in the given triangle.

Draw D E and D G perpendicular to B A and A C. In the triangles D E B, D F B the angles D E B and D B E are equal to the angles D F B and D B F (const.), and the side D B is common to both, therefore the sides D E and D F are equal (XXVI, Book I.): in the same manner it can be demonstrated that the lines D G and D F are equal; therefore the three lines D E, D F, and D G are equal, and therefore the circle described from the centre D with the radius D F passes through the points E and G; and because the angles at F, E, and G are right, the lines B C, B A, and A C are tangents to the circle (XVI, Book III.), therefore the circle F E G is inscribed in the given triangle.

(418) It is assumed in the demonstration of the proposition, that the two bisectors of the angles B C of the triangle will meet at the same point. This, however, may be proved by showing that they make angles with B C which are together less than two right angles.

In this demonstration, and in various other places, Euclid assumes, that any point whose distance from the centre of a circle is equal to the radius, must be on the circle. See (22).

(419) If D A be drawn it will bisect the angle A. For E D and G D are equal, and A D the common side is opposite to right angles E and G, and therefore the triangles D A E and D A G are in every respect equal. Therefore the angles D A E and D A G are equal.

Hence the lines bisecting the three angles of a triangle intersect at the same point, and that point is the centre of the inscribed circle.

(420) The areas of the three triangles B D C, C D A, and A D B, are respectively equal to half the rectangles under the radius of the inscribed circle and the sides B C, C A, and B A of the given triangle. Hence the area of the given triangle is equal to the rectangle under the radius of the inscribed circle and the semiperimeter of the triangle.

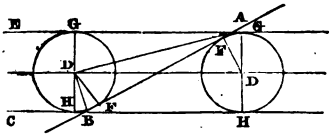
*** (421) Hence if the sides be given in numbers, the radius of the inscribed circle may be found by dividing the area (found by (186)) by the semiperimeter.

*** (422) The problem to inscribe a circle in a triangle is a particular case of a more general problem, 'To describe a circle touching three given right lines.'

1°. If the three given lines be parallel to one another, the problem is obviously impossible, since no circle touching two of them could touch the third.

2°. If two of the lines A G, B H be parallel and the third A B intersect them.

Draw the lines A D and B D bisecting the angles A and B. These will intersect, since they make angles with A B which are together less than two right angles. Let them meet at D. Perpendiculars D F, D G, and D H to the three given lines from D are equal. This may be proved as in the preceding proposition. Hence D is the centre and D F the radius of the circle.

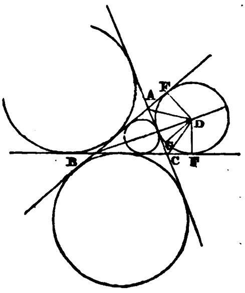


It appears from the diagram that there are two circles which touch the given right lines.

3°. Let the three given right lines intersect so as to form a triangle.

In this case the circle is determined as in the proposition. But this is not the only circle which may be drawn touching the given right lines. Draw the lines C D and A D bisecting the external angles at A and C. These, as before, will meet at D, and perpendiculars D E, D F, D G on the given lines from this point are equal. Hence D is the centre and D F the radius of a circle touching the three given lines. The demonstration of this is the same exactly as that of the proposition.

In the same manner two other circles may be described touching the given right lines as in the diagram.



Thus if three right lines intersect so as to form a triangle, four different circles may be described each touching them all.

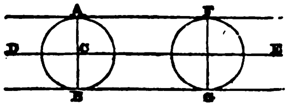
By this case it appears that the bisector of any internal angle of a triangle, and those of the remaining external angles, intersect at the same point.

4°. If the three given lines intersect at the same point, no circle can be described touching them all.

*** (423) It is plain the problem to describe a circle touching two given right lines is indeterminate. We can however in this case determine the *locus* of its centre.

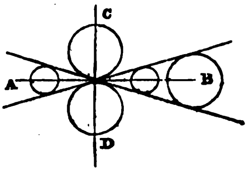
1°. If the two right lines be parallel.

Draw the line A B intersecting them perpendicularly, and bisect it at C, and through C draw D E parallel to the given lines. This will be the *locus* of the centres. For if any other perpendicular F G be drawn, a circle described on it as the diameter will touch the given lines.



2°. If the given lines intersect.

Draw the lines A B and C D bisecting the angles under the given lines. These lines will be the *locus* of the centres. The demonstrations will easily appear from that of Prop. IV. and from the annexed diagram.

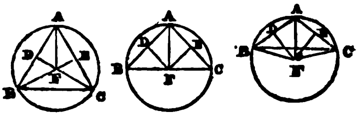


Proposition V. Problem.

(424) About a given triangle (B A C) to circumscribe a circle.

Bisect any two sides B A and A C of the given triangle, and through the points of bisection D and E draw D F and E F perpendicular to A B and A C, and from their point of concourse F draw to any angle A of the triangle B A C the line F A, the circle described from the centre F with the radius F A is circumscribed about the given triangle.

Draw F B and F C; in the triangles F D A, F D B the sides D A and D B are equal (const.), F D is common to both, and the angles at D are right, therefore the sides F A and F B are equal (IV, Book I.): in the same manner it can be demonstrated that the lines F A and F C are equal, therefore the three lines F A, F B, and F C are equal, and therefore the circle described from the centre F with the radius F A passes through B and C, and therefore is circumscribed about the given triangle B A C.



(425) Cor.—If the centre F fall within the triangle, it is evident all the angles are acute, for each of them is in a segment greater than a semicircle. If the centre F be in any side of the triangle the angle opposite to that side is right, because it is an angle in a semicircle (XXXI, Book III.); and if the centre fall without the triangle the angle opposite to the side which is nearest the centre is obtuse, because it is an angle in a segment greater than a semicircle (XXXI, Book III.).

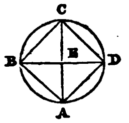
This problem has been anticipated in (78). It is, in fact, the same as to describe a circle through three given points which are not placed in the same right line.

(426) A perpendicular from F will evidently bisect B C, and therefore the perpendiculars from the middle points of the sides of a triangle have a common point of intersection, and this point is the centre of the circumscribed circle.

(427) It is assumed in the demonstration of this proposition, that the perpendiculars through D and E will intersect if produced. This may be proved by drawing the right line joining D and E. The perpendiculars evidently make with this line angles which are together less than two right angles.

PROPOSITION VI. PROBLEM.

(428) In a given circle (A B C D) to inscribe a square.



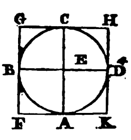
Draw any diameter A C of the given circle, draw B D perpendicular to it, and join A B, B C, C D, D A; A B C D is s square inscribed in the given circle.

Because the angles at E are right, and therefore equal, the arcs on which they stand are equal (XXVI, Book III.), and therefore their subtenses are equal (XXIX, Book III.); the figure A B C D is therefore equilateral; and because B D is a diameter, the angle B A D is in a semicircle, and therefore right (XXXI, Book III.); in the same manner it can be demonstrated that the angles B, C, and D are right; therefore, since the sides are also equal, the figure A B C D is a square.

The inscribed square is equal to twice the square of the radius, or to half the square of the diameter.

PROPOSITION VII. PROBLEM.

(429) About a given circle (A B C D) to circumscribe a square.



Draw any diameter A C of the given circle and B D perpendicular to it, and through their extremities A, B, C, and D, draw the lines K F, F G, G H, and H K tangents to the circle; the figure F G H K is a square circumscribed about the given circle.

Because E A is drawn from the centre to the point of contact the angle E A F is right (XVIII, Book III.), but the angle A E B is also right (const.), therefore the lines F K and B D are parallel; in the same manner it can be demonstrated G H is parallel to B D, and also that F G and K H are parallel to A C; therefore G D, B K, F C, and A H are parallelograms, and because the angles at A are right, the angles at G and H opposite to them are right (XXXIV, Book I.): in the same manner it can be demonstrated that K and F are right angles; therefore F G H K is a rectangle, and because A C and B D are equal, and F K and G A are equal to B D, and F G and K H are equal to A C, it is evident that F G H K is also equilateral, and therefore a square.

The circumscribed square is the square of the diameter, and is therefore twice the inscribed square, and four times the square of the radius.

PROPOSITION VIII. PROBLEM.

(430) In a given square (F G H K) to inscribe a circle.

Bisect two adjacent sides (Fig. Prop. VII.) G H and F G of the given square in C and B, through C draw C A parallel to either F G or K H, and through B draw B D parallel to either G H or F K; the circle described from the centre E with the radius E C is inscribed in the given square.

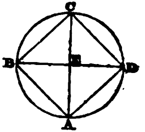
Because G E, E H, E K, and E F are parallelograms (const.), their opposite sides are equal (XXXIV, Book I.), therefore C E and E B are equal to G B and G C; but G B and G C are equal, for they are halves (const.) of the equal lines F G and G H, therefore C E and E B are equal; but E D and E A are equal to C E and E B, for they are equal to C H and B F the halves of G H and F G (const.), therefore the four lines E C, E B, E A, and E D are equal, and therefore the circle described from the centre E with the radius E C passes through B, A, and D; and because the angles at C, B, A, and D are right, the sides of the square are tangents to the circle (XVI, Book III.), which is therefore inscribed in the given square.

PROPOSITION IX. PROBLEM.

(431) About a given square (A B C D) to circumscribe a circle.

Draw A C and B D intersecting one another in E; the circle described from the centre E with the radius E A must pass through B C and D.

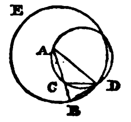
For since A B C is an isosceles triangle, and the angle B is right, the other angles are each half a right angle (XXXII, Book I.); in the same manner it can be demonstrated that each of the angles into which the angles of the square are divided, is half a right angle, they are therefore all equal, and therefore in the triangle A E B as the angles E A B and E B A are equal, the sides E A and E B are equal (VI, Book I.); in the same manner it can be demonstrated that E D and E C are equal to E A and E B, therefore the four lines E A, E B, E C, and E D are equal, and therefore the circle described from the centre E with the radius E A passes through B, C, and D, and is circumscribed about the given square.



PROPOSITION X. PROBLEM.

(432) To construct an isosceles triangle, in which each of the angles at the base shall be double of the vertical angle.

Take any line A B and divide it in C, so that the rectangle under A B and C B shall be equal to the square of A C (XI, Book II.); from the centre A with the radius A B describe a circle B E D, and inscribe in it a line B D equal to A C (I); join A D, and B A D is an isosceles triangle, in which the angles B and D are each double of the angle A.

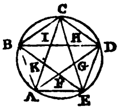


Draw D C, and circumscribe a circle A C D about the triangle D C A.

Because the rectangle under A B and B C is equal to the square of A C (const.), or to the square of B D (const.), the line B D is a tangent to the circle A C D (XXXVII, Book III.), therefore the angle B D C is equal to the angle A in the alternate segment (XXXII, Book III.); add to both the angle C D A, and B D A is equal to the sum of the angles C D A and A; but since the sides A B and A D are equal, the angles B and B D A are equal, therefore the angle B is equal to the sum of C D A and A; but the external angle B C D is equal to the sum of C D A and A (XXXII, Book I.), therefore the angles B and B C D are equal, and therefore the sides B D and C D are equal (VI, Book I.); but B D and C A are equal (const.), therefore C D and C A are equal, and therefore the angles A and C D A are equal; but B D A is equal to the sum of the angles A and C D A, therefore it is double of A, and therefore the angle B is also double of A.

PROPOSITION XI. PROBLEM.

(433) In a given circle (A B C D E) to inscribe an equilateral and equiangular pentagon.



Construct an isosceles triangle, in which each of the angles at the base shall be double of the angle at the vertex (X), and inscribe in the given circle a triangle A C E equiangular to it (II); bisect the angles at the base A and E by the right lines A D and E B, and join A B, B C, C D, and D E.

Because each of the angles C A E and C E A is double of E C A (const.), and is bisected, the five angles C E B, B E A, A C E, C A D, and D A E are equal; and therefore the arcs upon which they stand are equal (XXVI, Book III.), and therefore the lines C B, B A, A E, E D, and D C which subtend these arcs are equal (XXIX, Book III.), and therefore the pentagon A B C D E is equilateral.

And because the arcs A B and D E are equal, if the arc B C D be added to both, the arc A B C D is equal to B C D E, and therefore the angles A E D and B A E standing upon them are equal (XXVII, Book III.), in the same manner it can be demonstrated that all the other angles are equal, and therefore the pentagon is also equiangular.

⁂ (434) Each diagonal of a regular pentagon is parallel to the side with which it is not conterminous. For since the arcs B A and D E are equal, the chords B D and A E are parallel (379); and the same may be applied to the other diagonals.

Since the arcs A B and D E are together equal to B C D, it follows (380) that the angle B F A is equal to B A D. Hence it appears that A B F is an isosceles triangle equiangular with A C E, and therefore having its base angle equal to twice its vertical angle. The same is true of the triangles B C K, G C D, and F D E.

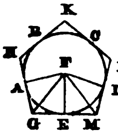
It is easy to see that the figure A B H E is a lozenge; and the same may be proved of A B C G, &c.

The figure F G H I K is a regular pentagon.

PROPOSITION XII. PROBLEM.

(435) About a given circle (A B C D E) to circumscribe an equilateral and equiangular pentagon.

Let the points A, B, C, D, and E be the vertices of the angles of an equilateral pentagon inscribed in the circle, and draw G H, H K, K L, L M, and M G tangents to the circle at these points; G H K L M is an equilateral and equiangular pentagon, circumscribed about the given circle.



Draw F A, F G, F E, F M, and F D. In the triangles F G A, F G E the sides G A and G E are equal (351), and also F A and F E; F G is common, therefore the angles F G A and F G E are equal, and also A F G and E F G, therefore the angle A G E is double of F G E, and A F E double of G F E; in the same manner it can be demonstrated that D M E is double of F M E, and that D F E is double of M F E; but since the arcs A E and E D are equal (const.), the angles A F E and D F E are equal (XXVII, Book III.), and therefore their halves G F E and M F E are equal, and the angles F E G and F E M are also equal, and the side E F is common, therefore the angles F G E and F M E are equal, and also the sides G E and E M, and therefore the line G M is double of G E; in the same manner it can be demonstrated that G H is double of G A, but G E and G A are equal, therefore G M and G H are equal; in the same manner it can be demonstrated that the other sides are equal, and therefore the pentagon G H K L M is equilateral; and because the angles D M E and A G E are double of F M E and F G E, and F M E and F G E are equal, D M E is equal to A G E; and in the same manner it can be demonstrated that the other angles are equal, and therefore G H K L M is also equiangular.

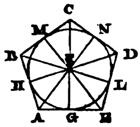
Proposition XIII. Problem.

(436) In a given equilateral and equiangular pentagon (A B C D E) to inscribe a circle.

Bisect any two adjacent angles A and E by the right line A F and E F, and from their point of concourse F draw F G perpendicular to A E, the circle described from the centre F with the radius F G is inscribed in the given pentagon.

Draw F B, F C, and F D, and from F let fall the perpendiculars F H, F N, F M, F L.

In the triangles A F B, A F E the sides A B and A E are equal (hyp.), A F is common, and the angles F A B and F A E are equal



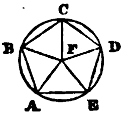
(const.), therefore the angles A B F and A E F are equal (IV, Book I.); but the angles A B C and A E D are also equal (hyp.), therefore since A E F is half of A E D (const.), A B F is half of A B C; in the same manner it can be demonstrated that the other angles of the pentagon are bisected by the lines drawn from F: wherefore in the triangles F B H, F B M, the angles F B H and F B M are equal, the angles at H and M are right, and the side F B opposite to the equal angles H and M is common, therefore the sides F H and F M are equal (XXVI, Book I.); and in the same manner it is proved that all the perpendiculars are equal, therefore the circle described from the centre F with the radius F G passes through the points H, M, N, and L, and the sides of the given pentagon are tangents to it, because the angles at G, H, M, N, and L are right.

PROPOSITION XIV. PROBLEM.

(437) To circumscribe a circle about a given equilateral and equiangular pentagon (A B C D E).

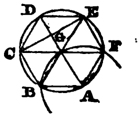
Bisect the angles A and E by the right lines A F and E F; the circle described from their point of concourse F as centre with the radius A F passes through the points B, C, D, and E.

Draw F B, F C, and F D. In the triangles F A E and F A B the sides F A and A E are equal to F A and A B, and the angle F A E is equal to F A B (const.), therefore the angles F B A and F E A are equal (IV, Book I.); but the angles A B C and A E D are also equal (hyp.); therefore since the angle F E A is half of A E D (const.), F B A is half of A B C, and therefore A B C is bisected by F B; and in the same manner it can be demonstrated that the angles C and D are bisected. Hence in the triangle A F E the angles F A E and F E A, being halves of the equal angles B A E and A E D, are equal, and therefore the sides F E and F A are equal (VI, Book I.); and in the same manner it is proved that the remaining lines F B, F C, and F D are equal; therefore the five lines F A, F B, F C, F D, and F E are equal, and therefore the circle described from the centre F with the radius F A passes through the points B, C, D, and E, and therefore is circumscribed about the given pentagon.



PROPOSITION XV. PROBLEM.

(438) In a given circle (A B C D E F) to inscribe an equilateral and equiangular hexagon.



Let G be the centre of the given circle; draw any diameter A G D; from the centre A with the radius A G describe a circle, and, from its intersections B and F with the given circle, draw the diameters B E and F C; join A B, B C, C D, D E, E F, and F A, and the figure A B C D E F is an equilateral and equiangular hexagon inscribed in the given circle.

Since the lines A B and A G are equal, as being radii of the same circle B G F, and G A and G B also equal, as being radii of the same circle A B C D E F, the triangle B G A is equilateral, and therefore the angle B G A is the third part of two right angles (XXXII, Book I.). In like manner it is proved that the triangle A G F is equilateral, and the angle A G F equal to one third part of two right angles; but the angles B G A and A G F together with F G E are equal to two right angles (XIII, Book I.), therefore F G E is one third part of two right angles, and therefore the three angles B G A, A G F, and F G E are equal, and also the angles vertically opposite to them E G D, D G C, and C G B; hence the six angles at the centre G are equal, and therefore the arcs on which they stand are equal, and the lines subtending those arcs (XXIX, Book III.); therefore the hexagon A B C D E F is equilateral, and also, since each of its angles is double the angle of an equilateral triangle, it is equiangular.

(439) It may be proved in general that every equilateral figure inscribed in a circle must be equiangular, for its angles are contained in equal arcs, and therefore stand on equal arcs.

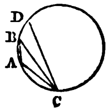
(440) The side of the regular hexagon is equal to the radius of its circumscribing circle, and its area is six times that of an equilateral triangle constructed on the radius of this circle.

If any three alternate angles A C E of the hexagon be joined by right lines, they will form the inscribed equilateral triangle.

PROPOSITION XVI. PROBLEM.

(441) In a given circle (C A D) to inscribe an equilateral and equiangular quindecagon.

Let C D be the side of an equilateral triangle inscribed in the circle C A D, and C A the side of an equilateral pentagon also inscribed in the circle C A D; bisect the arc A D; the right line joining A B is the side of the inscribed quindecagon. For if the whole circumference be divided into fifteen parts, the arc C D, since it is the third part of the whole circumference, contains five of these parts; in like manner the arc C A contains three of them, therefore the arc A D contains two, and therefore the arc A B is the fifteenth part of the whole circumference, and A B is the side of the inscribed equilateral quindecagon.



BOOK V.

DEFINITIONS.

(442). I. A less magnitude is said to be a part of a greater magnitude when the less measures the greater; that is, when the less is contained a certain number of times exactly in the greater.

The word 'part,' as applied in this definition, signifies an *aliquot* part or *submultiple.*

One quantity is said to measure another when, by continual subtraction of the former from the latter, a remainder is at length obtained equal to the former. In such a case it is plain that the former quantity multiplied by a certain integer number will become equal to the latter. Of two magnitudes thus related, the greater is said to be a *multiple* of the lesser, and the lesser is said to be a *submultiple* or *aliquot part* of the greater. Hence the meaning of the following definition is apparent.

(443) II. A greater magnitude is said to be a multiple of a less, when the greater is measured by the less, that is, when the greater contains the less a certain number of times exactly.

By the greater containing the less 'a certain number of times *exactly*, is meant, that the less is a *submultiple* of the greater, as already explained.

(444) A greater quantity is said to contain a lesser, as often as the lesser is capable of being successively subtracted from the greater. If the greater be not a multiple of the lesser, there will be a final remainder less than the lesser quantity. The number of times the lesser is contained in the greater is expressed by that integer by which the lesser must be multiplied, in order to obtain the highest multiple of it which is contained in the greater.

The student should be cautious not to confound the expressions 'measures' and 'is contained in.' The number 3 is 'contained three times in 10,' but does not measure 10, because there is a remainder 1 less than 3. Again, 3 is contained also three times in 11, but it does not measure it. On the other hand it 'measures' 9, being contained in it three times exactly without any remainder.

(445) It is evident that one quantity cannot be said to be contained in or to measure another, unless they be quantities of the *same kind*. Thus, for example, it would be manifestly absurd to say, that a line was contained a certain number of times in a surface, linear and superficial magnitude admitting of no comparison. No increase could render a line equal to a surface, because no increase could give it *breadth*, which is essential to a *surface*. In like manner the magnitude of a *surface* admits of no comparison with that of a *solid*, because no increase can give the one *thickness* which is essential to the other.

A line may be compared with a line, a surface with a surface, or a solid with a solid, as to magnitude, but none of these species can be compared with each other. This, however, does not apply to the lower species of magnitude. Different species of lines may be compared as to magnitude, because they all agree in having length only. Thus we can readily conceive a right line equal in length to a circular arc. The same applies to the different species of surfaces.

Two magnitudes A, B are said to be *equimultiples* of two others *a*, *b*, when *a* and *b* measure A and B respectively the same number of times. Thus the length *one foot* and the number 36 are *equimultiples* of the length one inch and the number 3; for an *inch* measures a *foot* twelve times, and 3 measures 36 also twelve times.

(446) III. Ratio is a mutual relation of two magnitudes of the same kind to one another, with respect to quantity.

This definition has been by some commentators considered to be obscure and useless, and on the other hand greatly extolled by others. It is hoped, however, that the preceding observations will render it intelligible. Ratio is, in fact, the relation between two magnitudes with respect to *magnitude only*, that is, excluding every other property which they may have. Thus a circular arc and a straight line may agree as to magnitude, although they may differ in every other respect. When their ratio is considered, the figures, position, &c. are totally neglected, and nothing but their abstract magnitudes or lengths are considered. In the same manner we may conceive a circular arc double or triple a straight line.

The two magnitudes between which ratio subsists are stated to be 'of the same kind,' because if they were 'of different kinds,' they would not admit of any comparison as to magnitude, as has been already explained.

Two magnitudes are said to be *equimultiples* of two others, when they are measured by those others the same number of times.

From this definition of ratio, nothing in mathematics has been deduced. Simson thinks that it is an interpolation of some unskilful editor. We think, however, with Playfair, that finding it necessary to use the word "ratio," Euclid thought that it was essential to that order and method for which geometry is so conspicuous, to give, in the proper place, a formal definition of the word. Its meaning appears more clearly from the fifth definition. This conjecture seems to be countenanced by the definitions of a straight line and a plane which stand in precisely the same predicament, no property of the line or plane being deduced from these definitions.

We may here remark generally, that although the definitions and propositions of the fifth book are expressed as if they applied only to magnitude, they are equally applicable to any other species of quantity. The student will find no difficulty in applying them to *number*, which is that species of quantity from which the clearest notions of proportion may be derived.

(447) IV. Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other.

In order to have a ratio they must be 'of the same kind,' and being so, one admits of being increased by multiplication so as to exceed the other.

[The student is advised to commence the propositions of the fifth book, omitting this and the succeeding definitions, and only to read them as he shall find them referred to from the propositions.]

(448) V. If there be four magnitudes, and any equimultiples whatsoever of the first and third be taken, and also any equimultiples whatsoever of the second and fourth be assumed; if, according as the multiple of the first is greater, equal to, or less than the multiple of the second, the multiple of the third is also greater, equal to, or less than the multiple of the fourth; then the first is said to have to the second the same ratio as the third has to the fourth.

The two magnitudes between which a ratio is conceived to subsist, are called the 'terms' of the ratio. That which is taken first in expressing the ratio is called the 'antecedent,' and the other is called the 'consequent.' To express the ratio both these terms are used, and the sign : is commonly placed between them. Thus if A be the antecedent, and B the consequent, the ratio is expressed by A : B. The ratio of A to B is also more compendiously expressed thus $\frac{A}{B}$

It is evident from all that has been observed, that a ratio depends on the *relative* and not on the *absolute* magnitudes of its terms, and that therefore, although the terms be changed, it is possible that the ratio may remain the same. In other words, the same ratio may subsist between different pairs of magnitudes. The object of the preceding definition is to establish a criterion by which two ratios may be determined to be equal, and the selection of a proper criterion for this equality has given rise to much discussion among geometers. Without entering into the metaphysics of this subject, we shall endeavour to unfold to the student the nature of ratio, by commencing with particular cases which do not present the difficulties which are sought to be removed by the general definition.

If there be two ratios $\frac{A}{B}$ and $\frac{a}{b}$, the question is, under what circumstances they will be equal?

If B be a multiple of A, it is plain that the ratios cannot be equal, unless b be an equal multiple of a. That is, if A be contained in B a certain number of times exactly, then a must be contained in b the same number of times exactly. Thus *ratios are the same if their consequents be equimultiples of their antecedents.*

In the same manner it will easily appear that they are also the same if *their antecedents be equimultiples of their consequents.*

These conditions may also be expressed thus: *If the antecedents of two ratios be equimultiples or equisubmultiples of their consequents the ratios are equal.*

If, however, neither term of the ratio be a multiple of the other, this test of equality of ratios will not be applicable. In that case let us suppose that some one magnitude M is found, which is at the same time a multiple of both terms of the ratio A : B, and let m be a magnitude which is the same multiple of a as M is of A, so that A and a are contained the same number of times in M and m respectively without remainders. In that case if M and m be equimultiples of B and b, the ratios A : B and a : b are equal, but otherwise not. Hence we perceive that 'if any equimultiples whatever of the antecedents of two ratios be also equimultiples of their consequents the ratios are equal.' It will be easily seen that this criterion is more general than the former, and includes it. We presume that, with very little attention, the student will perceive that, in these cases, the *relative magnitudes* of the terms of the two ratios must be necessarily the same.

If all ratios could be brought under the conditions just mentioned, there would have been no difficulty in the selection of a criterion for their equality. It however happens frequently that no magnitude can be found which is *at the same time a multiple of both terms of the same ratio.* In this case the criterion which we have just mentioned becomes quite inapplicable; and it is this which creates the greatest difficulty in the elementary theory of proportion. There are in this case no equimultiples of the antecedents which are also equimultiples of the consequents. Euclid has, however, instituted a criterion very analogous to that which we have explained in the other cases. Let M and m be equimultiples of A and a. Let the greatest multiple of B which is contained in M, and the greatest multiple of b which is contained in m, be found, and let us suppose that these are equimultiples of B and b. In this case it is evident that all equimultiples of B and b are either both greater or both less than M and m. Now if this be the case, whatever equimultiples of A and a, M and m may be, the ratios A : B and a : b are equal, and not otherwise.

A ratio is said to be *of major or minor inequality* according as the antecedent is greater or less than the consequent, and when they are equal it is a *ratio of equality.*

Euclid's criterion for the equality of two ratios may then be expressed thus: *Two ratios are equal when the ratios of every pair of equimultiples of their antecedents to every pair of equimultiples of their consequents are ratios of the same species of inequality.* Thus if A and *a* be multiplied by the same number, the results must be either both greater, equal to, or less than the results obtained by multiplying B and *b* by any number.

The present Bishop of Ferns, (Dr. Elrington,) in his edition of Euclid's Elements, published for the use of the students in the University of Dublin, has preferred to determine the equality of ratios by the *equisubmultiples* of the antecedents. His criterion is, that *two ratios are equal when every pair of equisubmultiples whatever of their antecedents are contained the same number of times in their respective consequents.*

It should be observed, that the definition of Euclid would be more correct if instead of the word 'any' the word 'every' were substituted. For as the text now stands it might be understood to be sufficient to establish the equality of the ratios if any one pair of equimultiples of the antecedents were found to fulfil the proposed condition, whereas this might happen with unequal ratios. It is necessary that the condition expressed in the definition should not only be fulfilled by *one* pair of equimultiples of the antecedents, but by *every* pair of equimultiples of them.

(449) VI. Magnitudes which have the same ratio are called proportionals. 'N. B. When four magnitudes are proportionals, it is usually expressed by saying, the first is to the second as the third to the fourth.'

The equality of two ratios is expressed by the sign :: or = interposed between them thus, $A : B :: a : b$ or $A : B = a : b$, or more shortly, $\frac{A}{B} = \frac{a}{b}$.

(450) VII. When of the equimultiples of four magnitudes (taken as in the fifth definition), the multiple of the first is greater than that of the second, but the multiple of the third is not greater than the multiple of the fourth; then the first is said to have to the second a greater ratio than the third magnitude has to the fourth; and, on the contrary, the third is said to have to the fourth a less ratio than the first has to the second.

(451) VIII. 'Analogy or proportion is the similitude of ratios.'

For the word 'similitude' here, 'equality' would be substituted with advantage.

(452) IX. Proportion consists in three terms at least.

This is not a definition but an inference; for since proportion is the equality of ratios, and equality implies at least two things, it follows, that in every proportion there must be at least two ratios. Each of these ratios must have two terms, and even if one of the two terms be the same in both, still there will be three terms in all.

(453) X. When three magnitudes are proportionals, the first is said to have to the third the duplicate ratio of that which it has to the second.

(454) The proportion in this case is said to be *continued*. Any number of magnitudes are said to be in continued proportion when the ratios of every successive pair of them are equal. Thus A, B, C, D are in continued proportion if the ratios $\frac{A}{B}$, $\frac{B}{C}$, and $\frac{C}{D}$ are equal; and this continued proportion is thus expressed, A : B : C : D.

When a series of quantities is in continued proportion, the first and last are called *extremes*, and the intermediate terms are called *means*.

Thus a *mean proportional* between two magnitudes is a third magnitude, such that, if it were placed between the other two, a series of three continued proportionals would be formed.

Two mean proportionals between two magnitudes are two magnitudes which, if interposed between the other two, would form a series of four continued proportionals, and so on.

When three magnitudes are in continued proportion, the third is called a *third proportional* to the other two.

(455) XI. When four magnitudes are continual proportionals, the first is said to have to the fourth the triplicate ratio of that which it has to the second, and so on, quadruplicate, &c. increasing the denomination still by unity, in any number of proportionals.

The ratio $\frac{A}{B}$ is duplicate of $\frac{a}{b}$, when it is equal to the ratio of a to a third proportional to a and b.

The ratio $\frac{A}{B}$ is triplicate of $\frac{a}{b}$, when it is equal to the ratio of a to a fourth continued proportional to a and b.

(456) The terms *subduplicate* and *subtriplicate* are sometimes used in geometry.

The ratio $\frac{A}{B}$ is *subduplicate* of $\frac{a}{b}$, when it is equal to the ratio of a to a mean proportional between a and b.

The ratio $\frac{A}{B}$ is said to be subtriplicate of $\frac{a}{b}$, when it is equal to the ratio of a to the first of two mean proportionals between a and b

Of Compound Ratio.

(457) XII. When there is any number of magnitudes of the same kind, the first is said to have to the last of them the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on to the last magnitude.

For example, if A, B, C, D be four magnitudes of the same kind, the first A is said to have to the last D the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the ratio of C to D; or, the ratio of A to D is said to be compounded of the ratios of A to B, B to C, and C to D.

And if A has to B the same ratio which E has to F; and B to C the same ratio that G has to H; and C to D the same that K has to L; then, by this definition, A is said to have to D the ratio compounded of ratios which are the same with the ratios of E to F, G to H, and K to L. And the same thing is to be understood when it is more briefly expressed by saying, A has to D the ratio compounded of the ratios of E to F, G to H, and K to L.

In like manner, the same things being supposed, if M has to N the same ratio which A has to D; then, for shortness' sake, M is said to have to N the ratio compounded of the ratios of E to F, G to H, and K to L.

The term *compound ratio*, like all other technical terms, is used for brevity, and to avoid circumlocution. A difficulty, however, arises with students respecting the use of this term, because it seems to imply something *more* than, or rather something different from what, it really is intended to express.

If we say that the ratio $\frac{A}{D}$ is compounded of the ratios $\frac{E}{F}$, $\frac{G}{H}$, and $\frac{K}{L}$, what is meant is this: if B and C be such magnitudes that $\frac{A}{B}=\frac{E}{F}$ and $\frac{B}{C}=\frac{G}{H}$, that in this case $\frac{C}{D}=\frac{K}{L}$.

(458) It is plain that the *duplicate ratio* is a ratio compounded of *two* equal ratios, and the *triplicate ratio* one compounded of *three* equal ratios.

(459) XIII. In proportionals, the antecedent terms are called homologous to one another, as also the consequents to one another.

'Geometers make use of the following technical words, to signify certain ways of changing either the order or magnitude of proportionals, so that they continue still to be proportionals.'

(460) XIV. Permutando, or alternando, by permutation or alternation. This word is used when there are four proportionals, and it is inferred that the first has the same ratio to the third which the second has to the fourth; or that the first is to the third as the second to the fourth: as is shown in XVI, Book V.

Permutation or *alternation* consists in the transposition of the means in four proportionals. Thus from $\frac{A}{B} = \frac{a}{b}$ we infer $\frac{A}{a} = \frac{B}{b}$.

(461) XV. Invertendo, by inversion; when there are four proportionals, and it is inferred that the second is to the first as the fourth to the third. Proposition B. Book V.

Inversion consists in the transposition of the antecedents and consequents. Thus from $\frac{A}{B} = \frac{a}{b}$ we infer $\frac{B}{A} = \frac{b}{a}$.

That ratio which is formed by the transposition of the terms of another ratio, is called the *reciprocal* of that other ratio. Thus *inversion* may be said to consist in changing the two ratios of a proportion into their reciprocals.

(462) It will appear hereafter, that the four terms of a proportion may be submitted to any change whatever in their order, provided that if one of the means be changed into an extreme, the other be also placed as the other extreme, and that if one of the extremes be placed as a mean, the other extreme be placed as the other mean. In other words, it is necessary either that the same terms remain as means and extremes, or that the means should be made extremes, and the extremes, means. Any change whatever in the places of the terms may be made, provided these conditions be observed, but not otherwise. This will be proved hereafter. It appears therefore that alternation and inversion are only two of a number of changes to which four proportionals may be submitted.

(463) XVI. Componendo, by composition; when there are four proportionals, and it is inferred that the first together with the second is to the second, as the third together with the fourth is to the fourth. XVIII, Book V.

Composition consists in substituting for the antecedents the sums of themselves and the consequents. Thus from $A : B = a : b$ we infer $A + B : B = a + b : b$, or from $\frac{A}{B} = \frac{a}{b}$ we infer $\frac{A+B}{B} = \frac{a+b}{b}$.

The sign $+$, or *plus*, between two or more quantities, implies *addition*.

(464) XVII. Dividendo, by division· when there are four proportionals, and it is inferred that the excess of the first above the second is to the second, as the excess of the third above the fourth is to the fourth. XVII, Book V.

Division consists in substituting for the antecedents the differences between themselves and the consequents. Thus from $\frac{A}{B} = \frac{a}{b}$ we infer $\frac{A-B}{B} = \frac{a-b}{b}$.

The sign $-$, or *minus*, between two quantities, implies the subtraction of the latter from the former.

(465) XVIII. Convertendo, by conversion ; when there are four proportionals, and it is inferred that the first is to its excess above the second as the third to its excess above the fourth. Proposition E. Book V.

Conversion consists in substituting the differences of the antecedents and consequents for the consequents. Thus from $\frac{A}{B} = \frac{a}{b}$ we infer $\frac{A}{A-B} = \frac{a}{a-b}$.

(466) It will appear that we may, in like manner, substitute the sums of the antecedents and consequents for the consequents.

(467) XIX. Ex æquali (sc. distantiâ), or ex æquo, from equality of distance: when there is any number of magnitudes more than two, and as many others, such that they are proportionals when taken two and two of each rank, and it is inferred that the first is to the last of the first rank of magnitudes as the first is to the last of the others: 'Of this there are the two following kinds, which arise from the different order in which the magnitudes are taken, two and two.'

(468) XX. Ex æquali, from equality. This term is used simply by itself, when the first magnitude is to the second of the first rank as the first to the second of the other rank; and as the second is to the third of the first rank, so is the second to the third of the other; and so on in order: and the inference is as mentioned in the preceding definition; whence this is called ordinate proportion. It is demonstrated in the twenty-second proposition, Book V.

Thus if there be two series of four magnitudes,

$$\begin{matrix} A & B & C & D, \\ a & b & c & d, \end{matrix}$$

and we have severally the following proportions,

$$\frac{A}{B} = \frac{a}{b}, \frac{B}{C} = \frac{b}{c}, \frac{C}{D} = \frac{c}{d},$$

we infer that $$\frac{A}{D} = \frac{a}{d}.$$

(469) XXI. Ex æquali in proportione perturbatâ seu inordinatâ, from equality in perturbate or disorderly proportion.* This term is used when the first magnitude is to the second of the first rank as the last but one is to the last of the second rank; and as the second is to the third of the first rank, so is the last but two to the last but one of the second rank; and as the third is to the fourth of the first rank, so is the third from the last to the last but two of the second rank, and so on in a cross order: and the inference is as in the eighteenth definition. It is demonstrated in the twenty-third proposition of Book V.

Thus if there be two series of four magnitudes as before, and we have severally the following proportions,

$$\frac{A}{B} = \frac{c}{d}, \frac{B}{C} = \frac{b}{c}, \frac{C}{D} = \frac{a}{b},$$

we infer that $$\frac{A}{D} = \frac{a}{d}.$$

Both this and the former inference come under one general principle, *scil.* that ratios which are compounded of equal ratios are equal.

One ratio is said to be *sesquiplicate* of another when it is compounded of that other ratio and its subduplicate.

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* Archimedis de sphærâ et cylindro, Prop. 4, lib. 2.

AXIOMS.

(470) I. Equimultiples of the same, or of equal magnitudes, are equal to one another.

(471) II. Those magnitudes, of which the same or equal magnitudes are equimultiples, are equal to one another.

Or what is the same, equisubmultiples of the same, or equal magnitudes, are equal.

(472) III. A multiple of a greater magnitude is greater than the same multiple of a less.

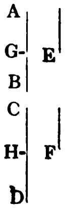
(473) IV. That magnitude, of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

Proposition I. Theorem.

(474) If any number of magnitudes be equimultiples of as many others, each of each: what multiple soever any one of the first is of its part, the same multiple shall all the first magnitudes taken together be of all the others taken together.

Let any number of magnitudes A B, C D be equimultiples of as many others E, F, each of each: whatsoever multiple A B is of E, the same multiple shall A B and C D together be of E and F together.

Because A B is the same multiple of E that C D is of F, as many magnitudes as there are in A B equal to E, so many are there in C D equal to F. Divide A B into magnitudes equal to E, *viz.* A G, G B; and C D into C H, H D, equal each of them to F: therefore the number of the magnitudes C H, H D, shall be equal to the number of the others A G, G B: and because A G is equal to E, and C H to F, therefore A G and C H together are equal to (Ax. II. Book I.) E and F together: for the same reason, because G B is equal to E, and H D to F, G B and H D together are equal to E and F together: wherefore as many magnitudes as there are in A B equal to E, so many are there in A B, C D together equal to E and F together: therefore, what-



soever multiple A B is of E, the same multiple is A B and C D together of E and F together.

Therefore, if any magnitudes, how many soever, be equimultiples of as many, each of each; whatsoever multiple any one of them is of its part, the same multiple shall all the first magnitudes be of all the others: 'For the same demonstration holds in any number of magnitudes, which was here applied to two.'

The reasoning in this book is employed about properties of magnitude in general, and therefore cannot be easily referred to or illustrated by diagrams. This renders the demonstrations in some degree confused and perplexed. Also the arithmetical relations which exist, or are instituted, between the quantities under consideration are expressed in ordinary language with so much prolixness, that the parts of a very simple demonstration become so separated one from another, that the student feels extreme difficulty in perceiving the steps of the reasoning. This difficulty is, however, if we may use the expression, *purely verbal.* If the ideas could be exhibited without the intervention of language, all difficulty would disappear. We shall, however, be able to render the demonstrations shorter and more easily intelligible by using, as in algebra, letters to express the quantities or magnitudes, and the usual symbols to express arithmetical operations. The demonstration of the first proposition may then be expressed as follows:—

Let A, B, and C be magnitudes which are equimultiples of A′, B′, C′. Suppose, for example, that the former are respectively three times the latter. We have the following equalities:—

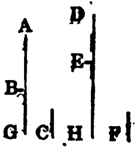
$$A = A' + A' + A',$$
$$B = B' + B' + B',$$
$$C = C' + C' + C'.$$

Adding these equals, we find that $A + B + C$ is three times $A' + B' + C'$, that is, $A + B + C$ is the same multiple of $A' + B' + C'$ as A, B, and C are respectively of A′, B′, and C′. The same would evidently be true if A, B, and C were supposed to be any other equimultiples of A′, B′, and C′.

Proposition II. Theorem.

(475) If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth; then shall the first together with the fifth be the same multiple of the second, that the third together with the sixth is of the fourth.

Let A B the first be the same multiple of C the second, that D E the third is of F the fourth; and B G the fifth the same multiple of C the second, that E H the sixth is of F the fourth: then shall A G, the first together with the fifth, be the same multiple of C the second, that D H, the third together with the sixth, is of F the fourth.

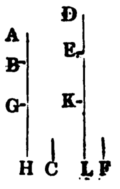


Because A B is the same multiple of C that D E is of F; there are as many magnitudes in A B equal to C, as there are in D E equal to F: in like manner, as many as there are in B G equal to C, so many are there in E H equal to F: therefore as many as there are in the whole A G equal to C, so many are there in the whole D H equal to F: therefore A G is in the same multiple of C that D H is of F; that is, A G, the first and fifth together, is the same multiple of the second C, that D H, the third and sixth together, is of the fourth F. If, therefore, the first be the same multiple, &c.

Otherwise thus:

Let the six quantities be A, B, C, D, E, F, and suppose, for example, that $A = 3B$ and $E = 2B$, it follows by adding these equals that $A + E = 5B$. Again, suppose that $C = 3D$ and $F = 2D$ it follows by adding these equals that $C + F = 5D$. Hence it follows that $A + E$ is the same multiple of B that $C + F$ is of D. And the same reasoning will apply if any other equimultiples be assumed.

COR.—'From this it is plain that if any number of magnitudes A B, B G, G H, be multiples of another C; and as many D E, E K, K L, be the same multiples of F, each of each: then the whole of the first, *viz.* A H, is the same multiple of C, that the whole of the last, *viz.* D L, is of F.'

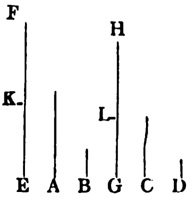


PROPOSITION III. THEOREM.

(476) If the first be the same multiple of the second, which the third is of the fourth; and if of the first and third there be taken equimultiples; these shall be equimultiples, the one of the second, and the other of the fourth.

Let A the first be the same multiple of B the second, that C the third is of D the fourth; and of A, C let equimultiples E F, G H be taken: then E F shall be the same multiple of B, that G H is of D.

Because E F is the same multiple of A that G H is of C, there are as many magnitudes in E F equal to A as there are in G H equal to C: let E F be divided into the magnitudes E K, K F, each equal to A; and G H into G L, L H, each equal to C: therefore the number of the magnitudes E K, K F, shall be equal to the number of the others G L, L H: and because A is the same multiple of B that C is of D, and that E K is equal to A, and G L equal to C; therefore E K is the same multiple of B that G L is of D: for the same reason, K F is the same multiple of B that L H is of D: and so, if there be more parts in E F, G H, equal to A, C: therefore, because the first E K is the same multiple of the second B, which the third G L is of the fourth D, and that the fifth K F is the same multiple of the second B, which the sixth L H is of the fourth D; E F, the first together with the fifth, is the same multiple (II, Book V.) of the second B, which G H, the third together with the sixth, is of the fourth D. If, therefore, the first, &c.

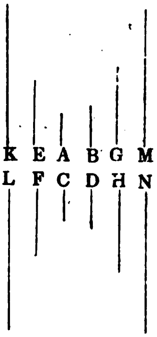


Otherwise thus:

Let A, B, C, D be the four quantities, and suppose, for example, that $A = 3B$ and $C = 3D$. Then $2A = 6B$ and $2C = 6D$; that is the equimultiples $2A$ and $2C$ of the first and third are also multiples of the second and fourth. The same reasoning is applicable in all cases.

PROPOSITION IV. THEOREM.

(477) If the first of four magnitudes has the same ratio to the second which the third has to the fourth; then any equimultiples whatever of the first and third shall have the same ratio to any equimultiples of the second and fourth, *viz.* 'the equimultiple of the first shall have the same ratio to that of the second, which the equimultiple of the third has to that of the fourth.'



Let A the first have to B the second the same ratio which the third C has to the fourth D; and of A and E let there be taken any equimultiples whatever C, F; and of B and D any equimultiples whatever G, H: then E shall have the same ratio to G which F has to H.

Take of E and F any equimultiples whatever K, L, and of G, H, any equimultiples whatever, M, N: then because E is the same multiple of A that F is of C; and of E and F have been taken equimultiples K, L; therefore K is the same multiple of A (III) that L is of C: for the same reason, M is the same multiple of B that N is of D. And because (hyp.), as A is to B, so is C to D, and of A and C have been taken certain equimultiples K, L, and of B and D have been taken certain equimultiples M, N; therefore if K be greater than M, L is greater than N; and if equal, equal; if less, less (Def. V.): but K, L are any equimultiples (const.) whatever of E, F, and M, N, any whatever of G, H; therefore as E is to G, so is (Def. V.) F to H.

Otherwise thus:

Let $A : B = C : D$, then $2\,A : 3\,B = 2\,C : 3\,D$. For all equimultiples of 2 A and 2 C are also equimultiples of A and C (III), and for the same reason all equimultiples of 3 B and 3 D are also equimultiples of B and D. But (Def. V.) A and C are either both greater, equal to, or less than B and D, and therefore any equimultiples of A and C are both greater, equal to, or less than B and D, and also greater, equal to, or less than any equimultiples of B and D (Ax. III.) Hence $2\,A : 3\,B = 2\,C : 3\,D$ (Def. V.): and the same reasoning is generally applicable.

Cor.—Likewise, if the first has the same ratio to the second, which the third has to the fourth, then also any equimultiples whatever of the first and third shall have the same ratio to the second and fourth: and in like manner, the first and the third shall have the same ratio to any equimultiples whatever of the second and fourth.

Let A the first have to B the second the same ratio which the third C has to the fourth D, and of A and C let E and F be any equimultiples whatever; then E shall be to B as F to D.

Take of E, F any equimultiples whatever K, L, and of B, D any equimultiples whatever G, H: then it may be demonstrated, as before, that K is the same multiple of A that L is of C: and because (hyp.) A is to B as C is to D, and of A and C certain equimultiples have been taken, *viz.* K and L; and of B and D certain equimultiples G, H; therefore if K be greater than G, L is greater than H; and if equal, equal; if less, less (Def. V.);

but K, L are any (const.) equimultiples whatever of E, F, and G, H any whatever of B, D; therefore as E is to B (Def. V.), so is F to D. And in the same way the other case is demonstrated.

Otherwise thus:

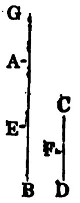
Let $A : B = C : D$, then $2\,A : B = 2\,C : D$, because all equimultiples of 2 A and 2 C are also equimultiples of A and C (III), and are therefore both either greater, equal to, or less than B and D (Def. V.) In the same manner it may be proved that $A : 2\,B = C : 2\,D$: and the same reasoning is generally applicable.

Proposition V. Theorem.

(478) **If one magnitude be the same multiple of another, which a magnitude taken from the first is of a magnitude taken from the other; the remainder is the same multiple of the remainder, that the whole is of the whole.**

Let the magnitude A B be the same multiple of C D, that A E taken from the first is of C F taken from the other: the remainder E B shall be the same multiple of the remainder F D, that the whole A B is of the whole C D.

Take A G the same multiple of F D that A E is of C F: therefore A E is (I.) the same multiple of C F that E G is of C D: but A E, by the hypothesis, is the same multiple of C F that A B is of C D; therefore E G is the same multiple of C D that A B is of C D; wherefore E G is equal (Ax. I.) to A B: take from each of them the common magnitude A E; and the remainder A G is equal to the remainder E B. Wherefore, since A E is the same multiple of C F (const.) that A G is of F D, and that A G is equal to E B; therefore A E is the same multiple of C F that E B is of F D: but A E is the same multiple of C F (hyp.) that A B is of C D; therefore E B is the same multiple of F D that A B is of C D.



Otherwise thus:

Let A, B, C, D be the quantities, and suppose that A and B are three times C and D respectively. We have then

$$A = B + B + B,$$
$$C = D + D + D.$$

Subtracting the latter from the former we find

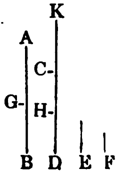
$$A - C = 3\,(B - D),$$

or that $A - C$ is three times $B - D$, that is, the same multiple of $B - D$ as A and C are of B and D respectively. The same reasoning is applicable in all cases.

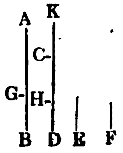
PROPOSITION VI. THEOREM.

(479) If two magnitudes be equimultiples of two others, and if equimultiples of these be taken from the first two; the remainders are either equal to these others, or equimultiples of them.

Let the two magnitudes A B, C D, be equimultiples of the two E, F, and let A G, C H taken from the first two be equimultiples of the same E, F: the remainders G B, H D shall be either equal to E, F, or equimultiples of them.



First, let G B be equal to E: H D shall be equal to F. Make C K equal to F: and because A G is the same multiple of E (hyp.) that C H is of F, and that G B is equal to E, and C K to F; therefore A B is the same multiple of E that K H is of F: but A B (hyp.) is the same multiple of E that C D is of F; therefore K H is the same multiple of F that C D is of F: wherefore K H is equal (Ax. I.) to C D: take away the common magnitude C H, then the remainder K C is equal to the remainder H D: but K C is equal (const.) to F: therefore H D is equal to F.



Next let G B be a multiple of E; H D shall be the same multiple of F. Make C K the same multiple of F that G B is of E: and because A G is the same multiple of E (hyp.) that C H is of F; and G B the same multiple of E that C K is of F; therefore A B is the same multiple of E (II) that K H is of F: but A B is the same multiple of E (hyp.) that C D is of F; therefore K H is the same multiple of F that C D is of F; wherefore K H is equal (Ax. I.) to C D: take away C H from both; therefore the remainder K C is equal to the remainder H D; and because G B is the same multiple of E (const.) that K C is of F, and that K C is equal to H D; therefore H D is the same multiple of F that G B is of E.

Otherwise thus:

Let A and B be the first two magnitudes, and C and D the others. Suppose that A = 5 C and B = 5 D. If the equimultiples of C and D, which are subtracted from A and B, be 4 C and 4 D, the remainders are evidently the magnitudes C and D themselves. If the equimultiples of C and D, thus subtracted, be any which are less than 4 C and 4 D, as 2 C and 2 D, the remainders are 3 C and 3 D, which are equimultiples.

By using m and n to express any integer numbers, m being supposed to be greater than n, this demonstration may be made general.

Let $A = m\,C$ and $B = m\,D$; since n is less than m, the equimultiples $n\,C$ and $n\,D$ are less than A and B. Let them be subtracted from the preceding equals and we have

$$A - n\,C = m\,C - n\,C = (m - n)\,C,$$
$$B - n\,D = m\,D - n\,D = (m - n)\,D.$$

If m exceed n by *one*, it is plain that these remainders are equal to C and D themselves, and if not, they are equimultiples obtained by multiplying C and D by $(m - n.)$

The next four Propositions are introduced by Simson.

Proposition A. Theorem.

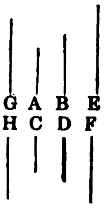
(480) If the first of four magnitudes have the same ratio to the second which the third has to the fourth; then, if the first be greater than the second, the third is also greater than the fourth; and if equal, equal; if less, less.

Take any equimultiples of each of them, as the doubles of each: then, by Def. V. of this book, if the double of the first be greater than the double of the second, the double of the third is greater than the double of the fourth: but if the first be greater than the second, the double of the first is greater than the double of the second; wherefore also the double of the third is greater than the double of the fourth; therefore the third is greater than the fourth: in like manner, if the first be equal to the second, or less than it, the third can be proved to be equal to the fourth, or less than it.

Proposition B. Theorem.

(481) If four magnitudes are proportionals, they are proportionals also when taken inversely.

Let $A : B = C : D$; then also, inversely, $B : A = D : C$.



Take of B and D any equimultiples whatever E and F; and of A and C any equimultiples whatever G and H. First, let E be greater than G, then G is less than E: and because (hyp.) A is to B as C is to D, and of A and C, the first and third, G and H are equimultiples; and of B and D, the second and fourth, E and F are equimultiples; and that G is less than E, therefore H is (Def. V.) less than F; that is, F is greater than H; if, therefore, E be greater than G, F is greater than H: in like manner, if E be equal to G, F may be shown to be equal to H; and if less, less; but E, F, are any equimultiples (const.) whatever of B and D, and G, H any whatever of A and C; therefore (Def. V.) as B is to A, so is D to C.

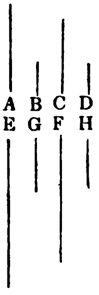
Otherwise thus:

Let m A and m C be any equimultiples of A and C, and n B, n D any equimultiples of B and D. Since A : B = C : D, m A and m C are either both greater, equal to, or less than n B and n D. If m A and m C be greater than n B and n D, then n B and n D are both less than m A and m C. If m A and m C be both less than n B and n D, then n B and n D will be both greater than m A and m C. Hence any equimultiples of B and D are both greater, equal to, or less than A and D, therefore B : A = D : C.

PROPOSITION C. THEOREM.

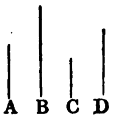
(482) If the first be the same multiple or submultiple of the second, that the third is of the fourth; the first is to the second as the third is to the fourth.

Let the first A be the same multiple of the second B that the third C is of the fourth D; A : B = C : D.



Take of A and C any equimultiples whatever E and F; and of B and D any equimultiples whatever G and H: then, because A is the same (hyp.) multiple of B that C is of D; and that E is the same (const.) multiple of A that F is of C; therefore E is the same multiple of B (III) that F is of D; that is, E and F are equimultiples of B and D: but G and H are equimultiples (const.) of B and D; therefore, if E be a greater multiple of B than G is of B, F is a greater multiple of D than H is of D; that is, if E be greater than G, F is greater than H: in like manner, if E be equal to G, or less than it, F may be shown to be equal to H, or less than it: but E, F are equimultiples (const.), any whatever, of A, C; and G, H, any equimultiples whatever of B, D; therefore (Def. V.) A : B = C : D.

Next, let the first A be the same submultiple of the second B that the third C is of the fourth D: A shall be to B as C is to D.

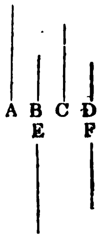


For since A is the same submultiple of B that C is of D, therefore B is the same multiple of A that D is of C: wherefore, by the preceding case, B : A = D : C; and therefore inversely, A : B = C : D.

PROPOSITION D. THEOREM.

(483) If the first be to the second as the third to the fourth, and if the first be a multiple or submultiple of the second; the third is the same multiple or submultiple of the fourth.

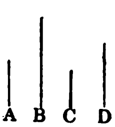
Let A : B = C : D; and first let A be a multiple of B: C shall be the same multiple of D.



Take E equal to A, and whatever multiple A or E is of B, make F the same multiple of D: then, because (hyp.) $A : B = C : D$; and of B the second, and D the fourth, equimultiples have been taken, E and F; therefore $A : E = C : F$; but A is equal (const.) to E, therefore C is equal (A) to F; and F is the same (const.) multiple of D that A is of B; therefore C is the same multiple of D that A is of B.

Next, let A be a submultiple of B; C shall be the same submultiple of D.

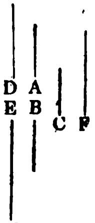
Because (hyp.) $A : B = C : D$; then inversely, $B : A = D : C$; but A is a submultiple (hyp.) of B, that is, B is a multiple of A; therefore, by the preceding case, D is the same multiple of C; that is, C is the same submultiple of D that A is of B.



Proposition VII. Theorem.

(484) Equal magnitudes have the same ratio to the same magnitude: and the same has the same ratio to equal magnitudes.

Let A and B be equal magnitudes, and C any other. A and B shall each of them have the same ratio to C: and C shall have the same ratio to each of the magnitudes A and B.



Take of A and B any equimultiples whatever D and E, and of C any multiple whatever F: then, because D is the same (const.) multiple of A that E is of B, and that A is equal (hyp.) to B: therefore D is (Ax. I.) equal to E: therefore if D be greater than F, E is greater than F; and if equal, equal; if less, less; but D, E are any equimultiples of A, B (const.), and F is any multiple of C; therefore (Def. V.) $A : C = B : C$.

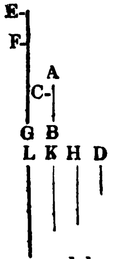
Likewise $C : A = C : B$. For, having made the same construction, D may in like manner be shown to be equal to E: therefore if F be greater than D, it is likewise greater than E; and if equal, equal; if less, less: but F is any multiple whatever of C, and D, E are any equimultiples whatever of A, B; therefore (Def. V.) $C : A = C : B$.

Proposition VIII. Theorem.

(485) Of two unequal magnitudes the greater has a greater ratio to any other magnitude than the less has: and the same magnitude has a greater ratio to the less of two other magnitudes than it has to the greater.

Let A B, B C be two unequal magnitudes, of which A B is the greater, and let D be any other magnitude. A B shall have a greater ratio to D than B C has to D: and D shall have a greater ratio to B C than it has to A B.

Fig. 1.



If the magnitude which is not the greater of the two A C, C B be not less than D, take E F, F G, the doubles of A C, C B, as in Fig. 1. But if that which is not the greater of the two A C, C B be less than D (as in Fig. 2 and 3), this magnitude can be multiplied, so as to become greater than D, whether it be A C or C B. Let it be multiplied until it become greater than D, and let the other be multiplied as often; and let E F be the multiple thus taken of A C, and F G the same multiple of C B: therefore E F and F G are each of them greater than D: and in every one of the cases, take H the double of D, K its triple, and so on, till the multiple of D be that which first becomes greater than F G: let L be that multiple of D which is first greater than F G, and K the multiple of D which is next less than L.

Then, because L is the multiple of D which is the first that becomes greater than F G, the next preceding multiple K is not greater than F G; that is, F G is not less than K: and since E F is the same multiple of A C (const.) that F G is of C B; therefore F G is the same multiple of C B (I, V.) that E G is of A B: that is, E G and F G are equimultiples of A B and C B: and since it was shown that F G is not less than K, and by the construction E F is greater than D; therefore the whole E G is greater than K and D together: but K together with D is equal (const.) to L; therefore E G is greater than L: but F G is not greater (const.) than L: and E G, F G were proved to be equimultiples of A B, B C; and L is a (const.) multiple of D; therefore (Def. VII.) A B has to D a greater ratio than B C has to D.

Fig. 2.

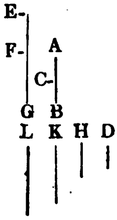
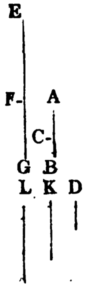


Fig. 3



Also D shall have to B C a greater ratio than it has to A B. For having made the same construction, it may be shown, in like manner, that L is greater than F G, but that it is not greater than E G: and L is a (const.) multiple of D; and F G, E G were proved to be equimultiples of C B, A B; therefore D has to C B a greater ratio (Def. VII.) than it has to A B.

Otherwise thus:

Let A + B and A be the two unequal magnitudes, and let C be any other. The ratio A + B : C is greater than A : C. Take equimultiples of A and B which are both greater than C, and let these be m A and m B; also take the least of those multiples of C which are greater than m A + m B, and let this be n C. Hence it follows that $(n-1)$ C is less than m A + m B, or than m (A + B); or what is the same, m (A + B) is greater than $(n-1)$ C or n C − C. But since n C is greater than m A + m B, and also C is less than m B, n C − C must be greater than m A, or m A is less than n C − C or than $(n-1)$ C. Hence m (A + B), which is a multiple of A + B, is greater than $(n-1)$ C, which is a multiple of C, while the same multiple m A of A is not greater than $(n-1)$ C. Therefore A + B : C is greater than A : C. (Def. VII.)

Also since $(n-1)$ C is greater than m A, but not greater than m (A + B), C : A is greater than C : A + B.

PROPOSITION IX. THEOREM.

(486) Magnitudes which have the same ratio to the same magnitude are equal to one another: and those to which the same magnitude has the same ratio are equal to one another.

Let A, B have each of them the same ratio to C: A is equal to B.

For, if they are not equal, one of them must be greater than the other: let A be the greater: then, by what was shown in the preceding proposition, there are some equimultiples of A and B, and some multiple of C, such that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than that of C. Let these multiples be taken; and let D, E be the equimultiples of A, B, and F the multiple of C, such that D may be greater than F, but E not greater than F: then, because A : C = B : C, and of A, B are taken equimultiples, D, E, and of C is taken a multiple F; and that D is greater than F; therefore E is also greater (Def. V.) than F: but E is not (const.) greater than F; which is impossible: therefore A and B are not unequal; that is, they are equal.

Next, let C have the same ratio to each of the magnitudes A and B: A shall be equal to B.

For if they are not equal, one of them must be greater than the other: let A be the greater: therefore, as was shown in Prop. VIII. there is some multiple F of C, and some equimultiples E and D of B and A, such that F is greater than E, but not greater than D: and because C : B = C : A, and that F the multiple of the first is greater than E the multiple of the second (Def. V.); therefore F the multiple of the third is greater than D the multiple of the fourth: but F is not (const.) greater than D; which is impossible.

Otherwise thus:

If A : C = B : C, then A = B. For if not, let A be greater than B, and let such equimultiples *m* A, *m* B of A and B be assumed, that while *m* A is greater than *n* C, *m* B is not greater than *n* C (VIII). Since A : C = B : C, all equimultiples of A and B must be at the same time greater, equal to, or less than *n* C; but *m* A and *m* B are equimultiples, one greater and the other less, which is absurd.

Again, if C : A = C : B; A = B. For *by inversion*, A : C = B : C; and therefore by the first case A = B.

PROPOSITION X. THEOREM.

(487) That magnitude which has a greater ratio than another has to the same magnitude, is the greater of the two: and that magnitude to which the same has a greater ratio than it has to another magnitude, is the lesser of the two.

Let A : C be greater than B : C. A is greater than B.

For, because A : C is greater than B : C, there are (Def. VII.) some equimultiples of A and B, and some multiple of C, such that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than it: let them be taken; and let D, E be the equimultiples of A, B, and F the multiple of C, such that D is greater than F; but E is not greater than F, therefore D is greater than E: and because D and E are equimultiples of A and B, and D is greater than E, therefore A is greater than B.

Next, let C : B be greater than A : B. B is less than A.

For (Def. VII.) there is some multiple F of C, and some equimultiples E and D of B and A, such that F is greater than E, but not greater than D: therefore E is less than D: and because E and D are equimultiples of B and A, and that E is less than D, therefore B is less than A.

PROPOSITION XI. THEOREM.

(488) Ratios that are equal to the same ratio are equal to one another.

Let A : B = C : D; and E : F = C : D: then A : B = E : F. Take of A, C, E any equimultiples whatever G, H, K; and of B, D, F any equimultiples whatever L, M, N. Therefore, since A : B = C : D, and G, H are taken equimultiples of A, C, and L, M, of B, D; if G be greater than L, H is greater than M; and if equal, equal; and if less, less. (Def. V.) Again, because E : F = C : D, and H, K are taken equimultiples of C, E; and M, N, of D, F; if H be greater than M, K is greater than N; and if equal, equal: and if less, less: but if G be greater than L, it has been shown that H is greater than M; and if equal, equal; and if less, less: therefore if G be greater than L, K is greater than N; and if equal, equal; and if less, less: and G, K are any equimultiples whatever of A, E; and L, N any whatever of B, F: therefore (Def. V.) A : B = E : F.

This proposition is to ratios what Axiom I. Book I. is to magnitudes.

Proposition XII. Theorem.

(489) If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so are all the antecedents taken together to all the consequents.

Let any number of magnitudes A, B, C, D, E, F, be proportionals; that is, A : B = C : D = E : F. Then A : B = A + C + E : B + D + F.

Take of A, C, E any equimultiples whatever G, H, K; and of B, D, F any equimultiples whatever L, M, N: then, because A : B = C : D = E : F; and that G, H, K are equimultiples of A, C, E, and L, M, N equimultiples of B, D, F; if G be greater than L, H is greater than M, and K greater than N; and if equal, equal; and if less, less (Def. V.): wherefore if G be greater than L, then G + H + K are greater than L + M + N; and if equal, equal; and if less, less: but G, and G + H + K are any equimultiples of A, and A + C + E; because if there be any number of magnitudes equimultiples of as many, each of each, whatever multiple one of them is of its part, the same multiple is the whole of the whole (I): for the same reason L, and L + M + N are any equimultiples of B, and B + D + F: therefore A : B = A + C + E : B + D + F.

PROPOSITION XIII. THEOREM.

(490) **If the first have to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth; the first also has to the second a greater ratio than the fifth has to the sixth.**

Let A the first have the same ratio to B the second which C the third has to D the fourth, but C the third a greater ratio to D the fourth, than E the fifth has to F the sixth: also the first A shall have to the second B a greater ratio than the fifth E has to the sixth F.

Because C has a greater ratio to D than E to F, there are some equimultiples of C and E, and some of D and F, such that the multiple of C is greater than the multiple of D, but the multiple of E is not greater than the multiple of F (Def. VII.): let these be taken, and let G, H be equimultiples of C, E, and K, L equimultiples of D, F, such that G may be greater than K, but H not greater than L: and whatever multiple G is of C, take M the same multiple of A; and whatever multiple K is of D, take N the same multiple of B: then, because A is to B (hyp.) as C to D, and of A and C, M and G are equimultiples; and of B and D, N and K are equimultiples; if M be greater than N, G is greater than K; and if equal, equal; and if less, less (Def. V.): but G is greater (const.) than K; therefore M is greater than N: but H is not (const.) greater than L: and M, H are equimultiples of A, E; and N, L equimultiples of B, F; therefore A has a greater ratio to B than E has to F (Def. VII.).

Otherwise thus:

Let A, B, C, D, E, F be six magnitudes, and let A : B = C : D, but C : D be greater than E : F, then A : B is greater than E : F. Since C : D is greater than E : F, equimultiples *m* C and *m* E of C and E may be assumed such that one is greater and the other less than equimultiples *n* D and *n* F of D and F. Now since A : B = C : D, if *m* C be greater than *n* D, *m* A will also be greater than *n* B. Hence *m* A is greater than *n* B, and *m* E less than *n* F. Therefore A : B is greater than E : F (Def. VII.)

This proposition is equivalent to stating that if any ratio be greater than another, every ratio which is equal to the former will also be greater than the latter.

It is evident also that if one ratio be greater than another, every ratio which is greater than the former is also greater than the latter.

Cor.—And if the first have a greater ratio to the second than the third has to the fourth, but the third the same ratio to the fourth which the fifth has to the sixth; it may be demonstrated, in like manner, that the first has a greater ratio to the second than the fifth has to the sixth.

Proposition XIV. Theorem.

(491) **If the first have the same ratio to the second which the third has to the fourth; then, if the first be greater than the third, the second shall be greater than the fourth; and if equal, equal; and if less, less.**

Let A : B = C : D; if A be greater than C, B is greater than D.

Because A is greater than C, and B is any other magnitude, A : B is greater than C : B (VII): but A : B = C : D (hyp.); therefore C : D is also greater than C : B (XIII): but of two magnitudes, that to which the same has the greater ratio is the lesser (X): therefore D is less than B; that is, B is greater than D.

Secondly, if A be equal to C, B is equal to D. For A : B = C : D or = A : D; therefore B is equal to D (IX).

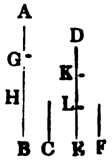
Thirdly, if A be less than C, B is less than D. For C is greater than A; and because C is to D as A is to B, therefore D is greater than B, by the first case; that is, B is less than D.

Proposition XV. Theorem.

(492) **Magnitudes have the same ratio to one another which their equimultiples have.**

Let A B be the same multiple of C that D E is of F: then C : F = A B : D E.

Because A B is the same multiple of C that D E is of F, there are as many magnitudes in A B equal to C as there are in D E equal to F: let A B be divided into magnitudes, each equal to C, *viz.* A G, G H, H B; and D E into magnitudes, each equal to F, *viz.* D K, K L, L E: then the number of the first A G, G H, H B is equal to the number of the last D K, K L, L E: and because A G, G H, H B are all equal, and that D K, K L, L E are also equal to one another; therefore (VII) A G : D K = G H : K L



= H B : L E : but as one of the antecedents to its consequent, (XII) so are all the antecedents together to all the consequents together, wherefore, as A G : D K = A B : D E : but A G is equal to C, and D K to F; therefore C : F = A B : D E.

Otherwise thus:

Let A, B be two magnitudes. A : B = A : B. Hence A : B = A + A : B + B (VII), or A : B = 2 A : 2 B. Hence (VII) A : B = A + 2 A : B + 2 B or A : B = 3 A : 3 B, and so on for all equimultiples of A and B.

PROPOSITION XVI. THEOREM.

(493) If four magnitudes *of the same kind* be proportionals, they are also proportionals when taken alternately.

Let A, B, C, D be four magnitudes of the same kind, and let A : B = C : D: they are also proportionals when taken alternately; that is, A : C = B : D.

Take of A and B any equimultiples whatever E and F; and of C and D take any equimultiples whatever G and H: and because E is the same multiple of A that F is of B, and that magnitudes have the same ratio to one another (XV) which their equimultiples have; therefore A : B = E : F: but A : B = C : D (hyp.); wherefore C : D = E : F (XI): again, because G, H are equimultiples of C, D, therefore C : D = G : H XV): but it was proved that C : D = E : F; therefore E : F = G : H (XI). But when four magnitudes are proportionals (XIV), if the first be greater than the third, the second is greater than the fourth; and if equal, equal; if less, less: therefore if E be greater than G, F likewise is greater than H; and if equal, equal; if less, less: and E, F are any (const.) equimultiples whatever of A, B; and G, H any whatever of C, D: therefore (Def. V.) A : C = B : D.

Otherwise thus:

Let A : B = C : D, and let m A and m B be any equimultiples of A and B, and let n C and n D be any equimultiples of C and D; m A : m B = A : B (XV), and therefore m A : m B = C : D (XI, and hyp.) Also n C : n D = C : D (XV), and therefore (XI) m A : m B = n C : n D. Hence m B is greater, equal to, or less than n D according as m A is greater, equal to, or less than n C (XIV). But m A and m B are any equimultiples of A and B, and n C and n D are any equimultiples of C and D. Hence, &c. (Def. V.)

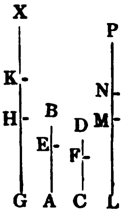
In this case it is necessary that the four magnitudes be of the same species, for otherwise, by alternation, ratios might be instituted between heterogeneous quantities (445).

PROPOSITION XVII. THEOREM.

(494) **If magnitudes, taken jointly, be proportionals, they are also proportionals when taken separately: that is, if two magnitudes together have to one of them the same ratio which two others have to one of these, the remaining one of the first two has to the other the same ratio which the remaining one of the last two has to the other of these.**

Let A B, E C, C D, D F be the magnitudes taken jointly which are proportionals: that is, A B : B E = C D : D F: they shall also be proportionals taken separately, *viz.* A E : E B = C F : F D.

Take of A E, E B, C F, F D any equimultiples whatever G H, H K, L M, M N; and again, of E B, F D take any equimultiples whatever K X, N P: and because G H is the same multiple of A E that H K is of E B, therefore G H is the same multiple (I) of A E that G K is of A B: but G H is the same multiple of A E that L M is of C F; therefore G K is the same multiple of A B that L M is of C F. Again, because L M is the same multiple of C F that M N is of F D; therefore L M is the same multiple (I) of C F that L N is of C D: but L M was shown to be the same multiple of C F that G K is of A B; therefore G K is the same multiple of A B that L N is of C D; that is, G K, L N are equimultiples of A B, C D. Next, because H K is the same multiple of E B that M N is of F D; and that K X is also the same multiple of E B that N P is of F D; therefore H X is the same multiple (II) of E B that M P is of F D. And because A B : B E = C D : D F (hyp.), and that of A B and C D, G K and L N are equimultiples, and of E B and F D, H X and M P are equimultiples; therefore (Def. V.) if G H be greater than H X, then L N is greater than M P; and if equal, equal; and if less, less: but if G H be greater than K X, then, by adding the common part H K to both, G K is greater than H X; wherefore also L N is greater than M P; and by taking away M N from both, L M is greater than N P: therefore if G H be greater than K X, L M is greater than N P. In like manner it may be demonstrated, that if G H be equal to K X, L M is equal to N P; and if less, less: but G H, L M are any equimultiples whatever of A E, C F (const.), and K X, N P are any whatever of E B, F D: therefore (Def. V.) as A E : E B = C F : F D.



Otherwise thus:

If $A + B : B = C + D : D$, then $A : B = C : D$. Take any multiples $m\,A$, $n\,B$ of A and B. First let $m\,A$ be greater than $n\,B$. Add $m\,B$ to both, and $m\,A + m\,B$ or $m\,(A + B)$ will be greater than $m\,B + n\,B$ or $(m + n)\,B$. But since $A + B : B = C + D : D$, it follows that if $m\,(A + B)$ be greater than $(m + n)\,B$, that $m\,(C + D)$ will also be greater than $(m + n)\,D$, or that $m\,C + m\,D$ will be greater than $m\,D + n\,D$. Take $m\,D$ from both, and $m\,C$ will be greater than $n\,D$; that is, if $m\,A$ be greater than $n\,B$, $m\,C$ will also be greater than $n\,D$.

In the same manner it may be proved, that if $m\,A = n\,B$, $m\,C = n\,D$, and that if $m\,A$ be less than $n\,B$, $m\,C$ will be less than $n\,D$. Hence (Def. V.) $A : B = C : D$ by division, (Def. XVII.)

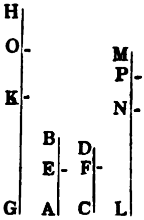
PROPOSITION XVIII. THEOREM.

(495) **If magnitudes, taken separately, be proportionals, they are also proportionals when taken jointly: that is, if the first be to the second as the third to the fourth, the first and second together are to the second as the third and fourth together to the fourth.**

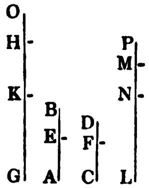
Let A E, E B, C F, F D be proportionals; that is, if $AE : EB = CF : FD$: then also $AB : BE = CD : DF$.

Take of A B, B E, C D, D F any equimultiples whatever G H, H K, L M, M N; and again, of B E, D F, take any equimultiples whatever K O, N P: and because K O, N P are equimultiples of B E, D F, and that K H, N M are likewise equimultiples of B E, D F; therefore if K O, the multiple of B E, be greater than K H, which is a multiple of the same B E, then N P, the multiple of D F, is also greater than N M, the multiple of the same D F; and if K O be equal to K H, N P is equal to N M; and if less, less.

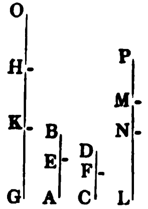
First, let K O be not greater than K H; therefore N P is not greater than N M: and because G H, H K are equimultiples of A B, B E, and that A B is greater than B E, therefore G H is greater (Ax. III.) than H K; but K O is not greater than K H; therefore G H is greater than K O. In like manner it may be shown that L M is greater than N P. Therefore if K O be not greater than K H, then G H, the multiple of A B, is always greater than K O, the multiple of B E; and likewise L M, the multiple of C D, is greater than N P, the multiple of D F.



Next, let K O be greater than K H; therefore, as has been shown, N P is greater than N M: and because the whole G H is the same multiple of the whole A B that H K is of B E, therefore the remainder G K is the same multiple of the remainder A E (V) that G H is of A B; which is the same that L M is of C D. In like manner, because L M is the same multiple of C D that M N is of D F, therefore the remainder L N is the same multiple of the remainder C F (V) that the whole L M is of the whole C D: but it was shown that L M is the same multiple of C D that G K is of A E; therefore G K is the same multiple of A E that L N is of C F; that is, G K, L N are equimultiples of A E, C F. And because K O, N P are equimultiples of B E, D F, therefore if from K O, N P there be taken K H, N M, which are likewise equimultiples of B E, D F, the remainders H O, M P are either equal to B E, D F, or equimultiples of them (VI). First, let H O, M P be equal to B E, D F: then because (hyp.) $AE : EB = CF : FD$, and that G K, L N are equimultiples of A E, C F; therefore $GK : EB = LN : FD$: but H O is equal to E P, and M P to F D; wherefore $GK : HO = LN : MP$: therefore if G K be greater than H O, L N is greater than (Ax. V.) M P; and if equal, equal; and if less, less.



But let H O, M P be equimultiples of E B, F D: then (hyp.) because $AE : EB = CF : FD$, and that of A E, C F are taken equimultiples G K, L N; and of E B, F D, the equimultiples H O, M P; if G K be greater than H O, L N is greater than M P; and if equal, equal; and if less, less (Def. V.); which was likewise shown in the preceding case. But if G H be greater than K O, taking K H from both, G K is greater than H O; wherefore also L N is greater than M P; and consequently adding N M to both, L M is greater than N P: therefore if G H be greater than K O, L M is greater than N P. In like manner it may be shown, that if G H be equal to K O, L M is equal to N P; and if less, less. And in the case in which K O is not greater than K H, it has been shown that G H is always greater than K O, and likewise L M greater than N P: but G H, L M are any equimultiples whatever of A B, C D (const.), and K O, N P are any whatever of B E, D F; therefore (Def. V.) $AB : BE = CD : DF$.



Otherwise thus:

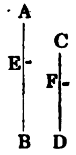
If $A : B = C : D$, then by composition $A + B : B = C + D : D$. For if $C + D : D$ be not equal to $A + B : B$, let $C + d : d$ be equal to $A + B : B$, d being a magnitude not equal to D. Since $A + B : B = C + d : d$, by (XVII), $A : B = C : d$; but (hyp.) $A : B = C : D$; therefore (XI) $C : D = C : d$, and therefore (IX) D and d are unequal, contrary to hyp.

PROPOSITION XIX. THEOREM.

(496) **If a whole magnitude be to a whole as a magnitude taken from the first is to a magnitude taken from the other; the remainder is to the remainder as the whole to the whole.**

Let the whole AB be to the whole CD as AE a magnitude taken from AB is to CF a magnitude taken from CD: the remainder EB shall be to the remainder FD as the whole AB to the whole CD.

Because $AB : CD = AE : CF$; therefore alternately (XVI), $AB : AE = CD : CF$: and because if magnitudes taken jointly be proportionals, they are also proportionals (XVII) when taken separately; therefore $BE : EA = DF : FC$; and alternately, $BE : DF = EA : FC$; but $AE : CF = AB : CD$ (hyp.); therefore also the remainder BE is to the remainder DF (XI) as the whole AB to the whole CD.



Otherwise thus:

If $A : B = C : D$, C and D being less than A and B, then $A - C : B - D = A : B$. For by alternation $A : C = B : D$, and by division $A - C : C = B - D : D$, and again by alternation $A - C : B - D = C : D = A : B$.

(497) COR.—If the whole be to the whole, as a magnitude taken from the first is to a magnitude taken from the other; the remainder shall likewise be to the remainder, as the magnitude taken from the first to that taken from the other. The demonstration is contained in the preceding.

Or, since $A - C : B - D = A : B$, and also $C : D = A : B$, therefore $A - C : B - D = C : D$.

The following proposition is added by Simson.

PROPOSITION E. THEOREM

(498) **If four magnitudes be proportionals, they are also proportionals by conversion: that is, the first is to its excess above the second as the third to its excess above the fourth.**

Let $A : B = C : D$, then $A : A - B = C : C - D$. For by division $A - B : B = C - D : D$, and by alternation $A - B : C - D = B : D$. But since $A : B = C : D$, by alternation $A : C = B : D$. Therefore

(XI) A : C = A − B : C − D, and by alternation A : A − B = C : C − D.

In a similar way it may be proved that A : A + B = C : C + D.

PROPOSITION XX. THEOREM.

(499) **If there be three magnitudes, and other three, which, taken two and two, have the same ratio; then if the first be greater than the third, the fourth is greater than the sixth; and if equal, equal; and if less, less.**

Let A, B, C,
A′, B′, C′,
be two series of three magnitudes, which taken two and two have the same ratio, *viz.*

$$A : B = A' : B', \qquad B : C = B' : C'.$$

First: let A be greater than C; A′ is also greater than C′.

Because A is greater than C, and B is any other magnitude, and that the greater has to the same magnitude a greater (VIII) ratio than the less has to it; therefore A : B is greater than C : B; but (hyp.) A′ : B′ = A : B; therefore (XIII) A′ : B′ is greater than C : B; and because B : C = B′ : C′, by inversion C : B = C′ : B′; and it was shown that A′ : B′ is greater than C : B; therefore A′ : B′ is greater than C′: B′; but the magnitude which has a greater ratio than another to the same magnitude, is the greater (X) of the two; therefore A′ is greater than C′.

Secondly, let A = C; then A′ = C′. Because A = C, A : B = C : B (VII); but (hyp.) A : B = A′: B′, and C : B = C′ : B′; wherefore A′ : B′ = C′ : B′ (XI); and therefore A′ = C′.

Thirdly, let A be less than C; A′ is also less than C′. For C is greater than A, and as was shown in the first case C : B = C′ : B′, and also B : A = B′ : A; therefore C′ is greater than A′ by the first case, or A′ is less than C′.

PROPOSITION XXI. THEOREM.

(500) **If there be three magnitudes, and other three, which have the same ratio taken two and two, but in a cross order; then if the first magnitude be greater than the third, the fourth is greater than the sixth; and if equal, equal; and if less, less.**

Let A, B, C,
A′, B′, C′,

be two series of three magnitudes which have the same ratio taken two and two, but in a cross order, *viz.*

$$A : B = B' : C', \quad B : C = A' : B'.$$

First, let A be greater than C; then A′ is also greater than C′. For since A is greater than C and B is any other magnitude, A : B is greater (VIII) than C : B; but (hyp.) B′ : C′ = A : B; therefore (XIII) B′ : C′ is greater than C : B; and because (hyp.) B : C = A′ : B′ by inversion, C : B = B′ : A′; and it was shown that B′ : C′ is greater than C : B; therefore B′ : C′ is greater than B′ : A′: but the magnitude to which the same has a greater ratio than it has to another, is the lesser (X) of the two: therefore C′ is less than A′; that is, A′ is greater than C′.

Secondly, let A = C; then also A′ = C′. Because A = C, A : B = C : B (VII): but A : B = B′ : C′ (hyp.): and C : B = A′ : B′; wherefore B′ : C′ = B′ : A′ (XI); and therefore A′ = C′, (IX).

Thirdly, let A be less than C; then also A′ is less than C′. For C is greater than A; and as was shown, C : B = B′ : A′, and also B : A = C′ : B′; therefore C′ is greater than A′ (by the first case); that is A′ is less than C′.

PROPOSITION XXII. THEOREM.

(501) If there be any number of magnitudes, and as many others, which taken two and two in order have the same ratio; the first has to the last of the first magnitudes the same ratio which the first has to the last of the others. N.B. *This is usually expressed by the words " ex æquali," or, " ex æquo."*

First, let A, B, C,
A′, B′, C′,

be two series of three magnitudes, which taken two and two have the same ratio, as expressed in Prop. XX.

Take any equimultiples a, a' whatever of A and A′, and also any equimultiples b, b' whatever of B, B′; and lastly, any equimultiples c, c' whatever of C, C′. Then because A : B = A′ : B′, and that a, a' are equimultiples of A, A′, and b, b' equimultiples of B, B′, therefore $a : b = a' : b'$; and for the same reason $b : c = b' : c'$ (IV); and because there are three magnitudes a, b, c, and other three a', b', c', which two and two have the same ratio

(XX); therefore a' is greater, equal to, or less than c', according as a is greater, equal to, or less than c; but a, a' are any equimultiples whatever of A, A′, and c, c' any equimultiples whatever of C, C′, therefore A : C = A′ : C′ (Def. V.)

Next let

A, B, C, D,
A′, B′, C′, D′,

be two series of four magnitudes, which taken two and two have the same ratio, *viz.*

$$A : B = A' : B',$$
$$B : C = B' : C',$$
$$C : D = C' : D',$$

then A : D = A′ : D′. Because taking the first three in each series, it follows from the first case that A : C = A′ : C′. Then there are two series of three magnitudes, *viz.*

A, C, D,
A′, C′, D′,

which taken two and two are in the same ratio, and therefore by the first case A : D = A′ : D′.

It is evident that this reasoning may be extended to series consisting of any number of magnitudes.

PROPOSITION XXIII. THEOREM.

(502) If there be any number of magnitudes, and as many others, which taken two and two in a cross order have the same ratio: the first has to the last of the first magnitudes the same ratio which the first has to the last of the others. N. B. *This is usually expressed by the words " ex æquali in proportione perturbatâ ;" or " ex æquo perturbato."*

First let

A, B, C,
A′, B′, C′,

be two series of three magnitudes related as explained in Prop. XXI.

Take any equimultiples whatever a, b and a' of A, B and A′: also any equimultiples whatever c, b', c' of C, B′, C′; and because a, b are equimultiples of A, B, and that magnitudes have the same ratio (XV) which their equimultiples have; therefore A : B = $a : b$, and in like manner B′ : C′ = $b' : c'$; but (hyp.) A : B = B′ : C′; therefore $a : b = b' : c'$ (XI); and because (hyp.) B : C = $a' : b'$, and b and a' are equimultiples of B and

A', and also c, b' of C, B'; therefore $b : c = a' : b'$ (IV); and it has been shown that $a : b = b' : c'$; therefore, because there are three magnitudes a, b, c, and other three a', b', c', which have the same ratio taken two aud two in a cross order, a' is greater, equal to, or less than c', according as a is greater, equal to, or less than c (XXI); but a, a' are any equimultiples whatever of A, A', and c c' any whatever of C, C'; therefore $A : C = A' : C'$, (Def. V.).

Secondly, let A, B, C, D,
A', B', C', D',
be two series of four magnitudes, which taken two and two in a cross order have the same ratio, *viz.*

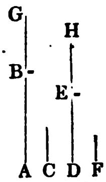
$$A : B = C' : D',$$
$$B : C = B' : C',$$
$$C : D = A' : B',$$

then also $A : D = A' : D'$. For the first three in the first series and the last three in the second, are two series of three magnitudes which are in the same ratio taken two and two in a cross order. Hence $A : C = B' : D'$ by the first case. Wherefore again by the first case $A : D = A' : D'$, and so on, whatever be the number of magnitudes.

PROPOSITION XXIV. THEOREM.

(503) If the first have to the second the same ratio which the third has to the fourth; and the fifth to the second, the same ratio which the sixth has to the fourth; the first and fifth together have to the second, the same ratio which the third and sixth together have to the fourth.

Let A B the first have to C the second the same ratio which D E the third has to F the fourth; and let B G the fifth have to C the second the same ratio which E H the sixth has to F the fourth: A G, the first and fifth together, have to C the second, the same ratio which D H, the third and sixth together, has to F the fourth.



Because $BG : C = EH : F$; by inversion, $C : BG = F : EH$: and because $AB : C = DE : F$ (hyp.); and $C : BG = F : EH$; *ex æquali* (XXII), $AB : BG = DE : EH$: and because these magnitudes are proportionals, they are likewise proportionals when taken (XVIII) jointly; therefore $AG : GB$

$= DH : HE$: but (hyp.) $GB : C = HE : F$; therefore *ex æquali* (XXII), $AG : C = DH : F$.

(504) Cor. 1.—If the same hypothesis be made as in the proposition, the excess of the first and fifth shall be to the second as the excess of the third and sixth to the fourth. The demonstration of this is the same with that of the proposition, if division be used instead of composition.

(505) Cor. 2.—The proposition holds true of two ranks of magnitudes, whatever be their number, of which each of the first rank has to the second magnitude the same ratio that the corresponding one of the second rank has to a fourth magnitude; as is manifest.

This proposition may be thus expressed, 'If two series of four proportionals have the same consequents, another series may be formed with the same consequents, and taking the sums of the antecedents as antecedents.'

Let the two series be $A : B = C : D$,
$A' : B = C' : D$,
then it follows that $A + A' : B = C + C' : D$.
Since $A' : B = C' : D$,
(*inv.*) $B : A' = D : C'$,
(*hyp.*) $A : B = C : D$,
(*ex. æq.*) $A : A' = C : C'$,
(*comp.*) $A + A' : A' = C + C' : C'$,
(*hyp.*) $A' : B = C' : D$,
(*ex. æq.*) $A + A' : B = C + C' : D$.

(506) In a similar way it may be proved that $A - A' : B = C - C' : D$.

(507) Also, it may be proved that if $A : B = C : D$, then $A + B : A - B = C + D : C - D$.

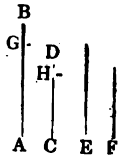
For,
(*comp.*) $A + B : B = C + D : D$,
(*div.*) $A - B : B = C - D : D$,
(*inv.*) $B : A - B = D : C - D$,
(*ex. æq.*) $A + B : A - B = (C + D) : C - D$.

Proposition XXV. Theorem.

(508) **If four magnitudes of the same kind be proportionals, the greatest and least of them together are greater than the other two together.**

Let the four magnitudes AB, CD, E, F be proportionals, *viz.* $AB : CD = E : F$; and let AB be the greatest of them, and consequently F the least (XIV). AB together with F shall be greater than CD together with E.

Take AG equal to E, and CH equal to F: then because $AB : CD = E : F$, and that AG is equal to E, and CH equal to F, therefore $AB : CD = AG : CH$ (VII, XI): and because AB the whole is to the whole CD as AG is to CH, likewise the remainder GB is to the remainder HD as the whole AB is to the whole (XIX) CD: but AB is greater (hyp.) than CD; therefore GB is greater than HD: and because AG is equal to E, and CH to F; AG and F together are equal to CH and E together: therefore if to the unequal magnitudes GB, HD, of which GB is the greater, there be added equal magnitudes, *viz.* to GB the two AG and F, and CH and E to HD; AB and F together are greater than CD and E.



The following propositions are added by Simson

PROPOSITION F. THEOREM.

(509) **Ratios which are compounded of the same ratios are equal to one another.**

Let $A : B = A' : B'$; and $B : C = B' : C'$: the ratio which is compounded of the ratios of $A : B$, and $B : C$, which by the definition of compound ratio is the ratio of $A : C$, shall be the same with the ratio of $A' : C'$, which, by the same definition, is compounded of the ratios of $A' : B'$, and B' to C'.

A.B.C.
A'.B'.C'.

Because there are three magnitudes, A, B, C, and three others A' B', C', which, taken two and two, in order, have the same ratio; *ex æquali* $A : C = A' : C'$ (XXII).

Next, let $A : B = B' : C'$, and $B : C = A' : B'$: therefore *ex æquali in proportione perturbatâ* (XXIII), $A : C = A' : C'$; that is, the ratio of $A : C$, which is compounded of the ratios of $A : B$, and $B : C$, is the same with the ratio of $A' : C'$, which is compounded of the ratios of $A' : B'$, and $B' : C'$. And in like manner the proposition may be demonstrated, whatever be the number of ratios in either case.

A.B.C.
A'.B'.C'.

PROPOSITION G. THEOREM.

(510) **If several ratios be equal to several other ratios, each to each; the ratio which is compounded of ratios which are equal to the first ratios, each to each, is equal to the ratio compounded of ratios which are equal to other ratios, each to each.**

In the two series of magnitudes

$$A, B, C, D,$$
$$A', B', C', D',$$

let $A : B = A' : B'$ and $C : D = C' : D'$. Also in the two series
$K, L, M,$
$K', L', M',$
let $K : L = A : B$ and $L : M = C : D$; also $K' : L' = A' : B'$, and $L' : M' = B' : C'$. Then the ratio $K : M$ is (by the definition of compounded ratio) compounded of the ratios of $K : L$ and $L : M$, which are equal to the ratios $A : B$ and $C : D$. Again, the ratio $K' : M'$ is compounded of the ratios $K' : L'$ and $L' : M'$ which are equal to $A' : B'$ and $C' : D'$. It is then to be proved that $K . M = K' : M'$.

Because $K : L = A : B$ and $A : B = A' : B'$, and $A' : B' = K' : L'$, therefore (XI) $K : L = K' : L'$. Again, because $L : M = C : D$, and $C : D = C' : D'$, and $C' : D' = L' : M'$, therefore (XI) $L : M = L' : M'$. Hence in the series K, L, M and K', L', M', (*ex. æq.*) $K : M = K' : M'$.

The student is advised to omit the remainder of this book.

PROPOSITION H. THEOREM.

(511) If a ratio which is compounded of several ratios be equal to a ratio which is compounded of several other ratios; and if one of the first ratios, or the ratio which is compounded of several of them, be equal to one of the last ratios, or to the ratio which is compounded of several of them; then the remaining ratio of the first, or, if there be more than one, the ratio compounded of the remaining ratios, are equal to the remaining ratio of the last, or, if there be more than one, to the ratio compounded of these remaining ratios.

Let the first ratios be those $A : B$, $B : C$, $C : D$, $D : E$, and $E : F$; and let the other ratios be $G : H$, $H : K$, $K : L$, and $L : M$: also, let the ratio $A : F$, which is compounded of (definition of compound ratio) the first ratios, be equal to $G : M$, which is compounded of the other ratios; and besides, let the ratio $A : D$, which is compounded of the ratios $A : B$, $B : C$, $C : D$, be equal to $G : K$, which is compounded of the ratios $G : H$, and $H : K$: then the ratio compounded of the remaining first ratios, to wit, $D : E$, and $E : F$, which compounded ratio is equal to $D : F$, shall be equal to $K : M$, which is compounded of the remaining ratios $K : L$, and $L : M$ of the other ratios.

A.B.C.D.E.F.
G.H.K.L.M.

Because, by the hypothesis, $A : D = G : K$, by inversion $D : A = K : G$; and (hyp.) $A : F = G : M$; therefore, (XXII), *ex æquali*, $D : F = K : M$.

PROPOSITION K. THEOREM.

(512) If there be any number of ratios, and any number of other ratios such that the ratio which is compounded of ratios which are equal to the first ratios, each to each, is equal to the ratio which is compounded of ratios which are equal, each to each, to the last ratios; and if one of the first ratios, or the ratio which is compounded of ratios which are equal to several of the first ratios, each to each, be equal to one of the last ratios, or to the ratio which is compounded of ratios which are equal, each to each, to several of the last ratios; then the remaining ratio of the first, or, if there be more than one, the ratio which is compounded of ratios which are equal each to each to the remaining ratio of the first, are equal to the remaining ratio of the last, or, if there be more than one, to the ratio which is compounded of ratios which are equal each to each to these remaining ratios.

Let the ratios $A : B$, $C : D$, $E : F$, be the first ratios; and the ratios $G : H$, $K : L$, $M : N$, $O : P$, $Q : R$, be the other ratios: and let $A : B = S : T$; and $C : D = T : V$; and $E : F = V : X$: therefore, by the definition of compound ratio, the ratio $S : X$ is compounded of the ratios $S : T$, $T : V$, and $V : X$, which are equal to the ratios $A : B$, $C : D$, $E : F$, each to each. Also, $G : H = Y : Z$; and $K : L = Z : a$; $M : N = a : b$; $O : P = b : C$; and $Q : R = c : d$: therefore, by the same definition, the ratio $Y : d$ is compounded of the ratios $Y : Z$, $Z : a$, $a : b$, $b : c$, and $c : d$, which are equal each to each, to the ratios $G : H$, $K : L$, $M : N$, $O : P$, and $Q : R$: therefore (hyp.) $S : X = Y : d$. Also, let the ratio $A : B$, that is, the ratio $S : T$, which is one of the first ratios, be equal to the ratio $e : g$, which is compounded of the ratios $e : f$, and $f : g$, which (hyp.) are equal to the ratios $G : H$, and $K : L$, two of the other ratios; and let the ratio $h : l$ be that which is compounded of the ratios $h : k$, and $k : l$, which are equal to the remaining first ratios, *viz.* $C : D$, and $E : F$; also, let the ratio $m : p$ be that which is compounded of the ratios $m : n$, $n : o$, and $o : p$, which are equal each to each, to the remaining other ratios, *viz.* $M : N$, $O : P$, and $Q : R$: then the ratio $h : l$ shall be equal to the ratio $m : p$; or $h : l = m : p$.

h, k, l.
A, B; C, D; E, F. S, T, V, X.
G, H; K, L; M, N; O, P; Q, R. Y, Z, *a, b, c, d.*
e, f, g. *m, n, o, p.*

$h, k, l.$
A, B; C, D; E, F. S, T, V, X.
G, H; K, L; M, N; O, P; Q, R. Y, Z, $a, b, c, d.$
$e, f, g.$ $m, n, o, p.$

Because $e : f = \mathrm{G} : \mathrm{H} = \mathrm{Y} : \mathrm{Z}$; and $f : g = \mathrm{K} : \mathrm{L} = \mathrm{Z} : a$ therefore, (XXII) *ex æquali*, $e : g = \mathrm{Y} : a$: and by the hypothesis, $\mathrm{A} : \mathrm{B} = \mathrm{S} : \mathrm{T} = e : g$; wherefore (XI) $\mathrm{S} : \mathrm{T} = \mathrm{Y} : a$; and, by inversion, $\mathrm{T} : \mathrm{S} = a : \mathrm{Y}$: but $\mathrm{S} : \mathrm{X} = \mathrm{Y} : d$; therefore, *ex æquali*, $\mathrm{T} : \mathrm{X} = a : d$; also (hyp.) because $h : k = \mathrm{C} : \mathrm{D} = \mathrm{T} : \mathrm{V}$; and $k : l = \mathrm{E} : \mathrm{F} = \mathrm{V} : \mathrm{X}$; therefore, *ex æquali*, $h : l = \mathrm{T} : \mathrm{X}$: in like manner it may be demonstrated that $m : p = a : d$; and it has been shown that $\mathrm{T} : \mathrm{X} = a : d$; therefore (XI) $h : l = m : p$. Q. E. D.

The propositions G and K are usually, for the sake of brevity, expressed in the same terms with propositions F and H: and therefore it was proper to show the true meaning of them when they are so expressed, especially since they are very frequently made use of by geometers.

BOOK VI.

DEFINITIONS.

(513) I. Similar rectilinear figures are those whose angles are severally equal each to each, and whose sides including equal angles are severally proportional.

(514) II. A straight line is said to be cut in extreme and mean ratio when the whole line is to one segment as that segment is to the remaining one.

This definition is thus expressed by Euclid: 'The whole line is to the greater segment as the greater segment is to the lesser.' But it is objectionable to assume in one part of a definition a property which may be deduced from the remainder of it. In this case, one of the segments is a mean proportional between the whole line and the other, and since the whole line is greater than the mean segment, so this mean segment must be greater than the other.

(515) III. The *altitude* of a figure is the perpendicular drawn from its vertex to its base.

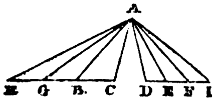
The altitude can scarcely be considered to have any distinct meaning, except as applied to a triangle, or a parallelogram, and the latter has no vertex properly speaking. The altitude of a parallelogram is the perpendicular drawn from its base to the opposite side.

It is evident that any side of a figure may be considered as the base.

Proposition I.

(516) Triangles and parallelograms having the same altitude are one to another as their bases.

Let the triangles be A B C, A E D, having a common vertex A, and their bases B C and E D on the same right line. Produce B E both ways, and take successively any number of parts B G, G H equal to B C; and E F, F I equal to D E;



and draw lines A G, A H, A F, A I from the common vertex A to their extremities.

Since the bases C B, B G, G H, are equal, the triangles on these bases are also equal (XXXVIII, Book I.), therefore the triangle A H C and its base H C are equimultiples of the triangle A B C and its base B C. In like manner it may be proved that the triangle D A I and its base D I are equimultiples of the triangle D A E and its base D E. It is evident that the triangle H A C is greater, equal to, or less than D A I, according as its base H C is greater, equal to, or less than the base D I. Hence it appears that since equimultiples of the first base and first triangle are at the same time greater, equal to, or less than equimultiples of the second base and second triangle, the triangles are as their bases.

Parallelograms having the same altitude are the doubles of triangles on their bases and having the same altitude, and are therefore proportional to them. But the triangles are as their bases (Part 1°.), and therefore their doubles (XV, Book V.), the parallelograms, are as their bases.

(517) Triangles and parallelograms having equal altitudes are as their bases.

For let the bases be placed on the same right line, and the triangles on the same side of it. The line joining their vertices will be then parallel to the base (XXXIII, Book I.), and the same demonstration may be applied as above.

The parallelograms are as their bases, being the doubles of triangles.

(518) Triangles and (their doubles) parallelograms having equal bases are as their altitudes.

For they are equal to right angled triangles or parallelograms, having bases and altitudes respectively equal; and in these latter the altitude may be taken as the base, and *vice versâ*. Hence the proof is reduced to (517).

(519) Triangles and parallelograms in general are in a ratio compounded of their bases and altitudes.

Let T and T' be two triangles or parallelograms, and let the base and altitude of the first be b, a and of the second b', a'. Let M be a triangle or parallelogram with the altitude a of the first, and base b' of the second. By (517) we have

$$T : M :: b : b',$$

and by (518) $M : T' :: a : a'$.

Hence $T : T' :: \left\{\begin{matrix} b : b' \\ a : a' \end{matrix}\right\}$*

That is, the triangles or parallelograms are in a ratio compounded of the ratios of their bases and altitudes.

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* We express a compounded ratio thus by enclosing the component ratios within a parenthesis.

(520) The rectangle under two lines is a mean proportional between their squares.

Let A and B be the lines. The square of A is to the rectangle A × B as A : B, since they have the same altitude A; and again A × B is to the square of B as A : B, since they have the same base B.

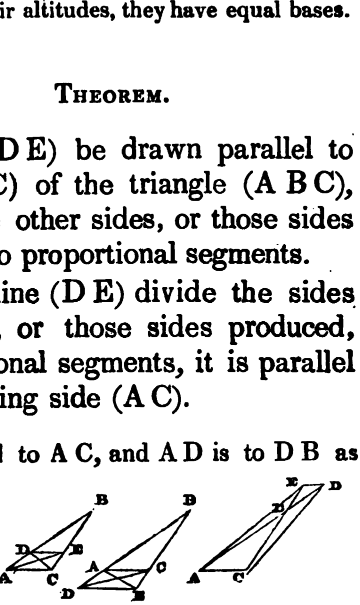
(521) If two triangles or parallelograms be as their bases, they have equal altitudes, and if they be as their altitudes, they have equal bases. These easily follow *ex absurdo*.

Proposition II. Theorem.

(522) 1°. If a right line (D E) be drawn parallel to any side (A C) of the triangle (A B C), it divides the other sides, or those sides produced, into proportional segments.

2°. And if a right line (D E) divide the sides of a triangle, or those sides produced, into proportional segments, it is parallel to the remaining side (A C).

Part 1°.—Let D E be parallel to A C, and A D is to D B as C E is to E B.



For draw A E and D C, and since the triangles E A D and E C D are upon the same base E D, and between the same parallels E D and C A, they are equal (XXXVII, Book I.), therefore A E D has the same ratio to D E B which C D E has to the same E D B; but A E D is to D E B as A D to D B (I), and C D E is to E D B as C E to E B (I), therefore A D is to D B as C E is to E B.

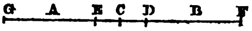
Part 2°.—Let A D be to D B as C E to E B, and the right line D E is parallel to A C.

Let the same construction remain, and A D is to D B as the triangle A E D to the triangle D E B (I), and as C E to E B, so is the triangle C D E to the triangle E D B (I); but A D is to D B as C E to E B (hyp.), therefore A E D is to D E B as C D E to the same E D B (VII, Book V.), therefore A E D is equal to C D E (IX, Book V.); but they are upon the same base D E, and at the same side of it, and therefore D E is parallel to A C (XXXIX, Book I.).

The enunciation of this proposition is inaccurate in several respects. In the first part the manner in which the parallel cuts the sides is not

distinctly described, and the second part is, strictly speaking, *false*, inasmuch as a line may cut two sides *proportionally*, and yet not be parallel to the third side.

To perceive these defects and the manner of correcting them, it is only necessary to consider in how many ways a finite right line may be divided in a given ratio. Let the line be A B, and let A D : D B be the given ratio, D will be one point of section such as required. Let C be the point of bisection, and take C D = C E, ∵ B E = A D, and A E = B D, ∵ B E : E A is the given ratio, and E is another point of section such as is required. Thus there are two points of internal section in a given ratio. In the same way if A F : B F and A G : B G be each the given ratio, F and G are two points of external section, which cut the line A B in the given ratio. It therefore appears that there are *four* points at which a line may be cut in a given ratio. Now it would be necessary, in order to render the first part of this proposition *distinct*, and the second part *true*, to state in the enunciation in which of these ways each side is cut.



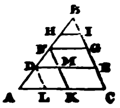
The enunciation would be correct if thus changed: '1°. If a line be drawn parallel to any side of a triangle, it divides the other sides, or those sides produced, so that their segments between the parallel and the third side shall have the same ratio to their segments between the parallel and the vertex of the opposite angle; and 2°. if a line cut the two sides in this manner it will be parallel to the third side.'

Dr. Elrington proposes to add to the present enunciation the words 'so that the homologous segments are at the same side' of the parallel or cutting line. But this, although less objectionable than Euclid's, is still imperfect, since it is only a distinction when the parallel cuts the sides *internally*. When it cuts them *externally*, all the segments lie at the same side of the cutting line, and therefore no distinction is thus introduced.

The enunciation of this important proposition would, however, be still further improved if given thus: If two indefinite intersecting right lines be cut by two parallel right lines three points of division are obtained on each line, *scil.* the point where they intersect each other, and their points of intersection with the parallels; the parts intercepted between any two of these points on one line are in the same ratio with the parts intercepted by the corresponding points on the other line, and if the points of section of the lines fulfil this condition, the lines joining them respectively will be parallel.

The three diagrams accompanying this proposition result from the three different ways in which the parallel may cut the sides: 1°. it may cut them internally: 2°. it may cut them produced through the base: 3°. it may cut them produced through the vertex.

In every case the two triangles which are proved equal are those which have the parallel (Part 1°.) or cutting line (Part 2°.) as their base, and their vertices at the extremities of the base of the given triangle. The common triangle with which these are compared is that which has the parallel (Part 1°.) or the cutting line (Part 2°.) as base, and its vertex at the vertex of the given triangle.



(523) If several parallels D E, F G, H I be drawn to the base of a triangle, every pair of corresponding segments in each side will be proportional. For draw D L, F K parallel to B C. In the parallelograms D C, F E, the opposite sides are equal, ∵ F M = E G, D L = E C. By the triangles F B G, A F K we have

B H : H F = B I : I G,

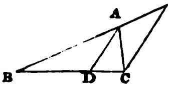
F D : D A = F M : M K = G E : E C, &c.

Proposition III. Theorem.

(524) 1°. A right line (A D) bisecting the angle of a triangle (B A C) divides the opposite sides into segments (B D, D C) proportional to the conterminous sides (B A, A C).

2°. And if a right line (A D) drawn from any angle of a triangle divide the opposite side (B C) into segments (B D, D C) proportional to the conterminous sides (B A, A C), it bisects the angle.

Part 1°.—Draw through C a right line C E parallel to A D until it meet the side B A produced to E.

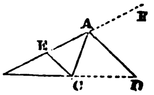


Because the lines A D and E C are parallel, the angle B A D is equal to the internal angle at the same side A E C (XXIX, Book I.), therefore the angle D A C is equal to A E C; but D A C is equal to the alternate angle A C E, therefore A C E and A E C are equal, and therefore the opposite sides A E and A C are equal (VI, Book I.); but since A D is parallel to E C, E A is to A B as C D is to D B; therefore since E A and A C are equal, A C is to A B as C D is to D B.

Part 2°.—Let the same construction remain, and B A is to A E as B D to D C (II); but B D is to D C as B A to A C (hyp.), therefore B A is to A E as B A to A C (XI, Book V.), and therefore A E and A C are equal (IX, Book V.), and the angle A E C is equal to A C E (V, Book I.); but since A D and E C are parallel, the angle D A C is equal to the alternate angle A C E, and the angle B A D equal to the internal angle at the same side A E C (XXIX, Book I.); therefore, since A E C and A C E are equal, B A D and D A C are also equal, and therefore the right line A D bisects the angle B A C,

(525) From this proposition it follows that if the same line bisect the base and vertical angle, the triangle is isosceles.

(526) This proposition is applicable also to the bisector of the external angle of the triangle. Let the side B A be produced to F, and draw A D bisecting the angle C A F, and through C draw C E parallel to A D. The angles A C E and A E C are proved equal to C A D and D A F, as in the proposition, and are therefore equal, and therefore A C = A E. In the triangle B E C, A D is a parallel to E C cutting the sides produced (II), $\therefore$ B A : A E = B D : D C, $\because$ B A : A C = B D : D C. Also, if B A : A C = B D : D C, A D will bisect the angle C A F. For by the parallels as before B A : A E = B D : D C, $\because$ A E = AC, $\therefore$ A C E = A E C; but these angles are respectively equal to C A D and D A F.



If the triangle be isosceles the bisector of the external angle is parallel to the base. Even in this case the proportionality of the external segments to the sides is preserved, for the point of external section becomes, as it were, infinitely distant, and the infinite segments whose difference is the base are equal, since their difference bears an infinitely small ratio to the segments themselves.

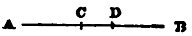
(527) The segments of the base made by the external bisector are proportional to those made by the internal bisector, since each is proportional to the sides of the triangle. Hence the bisectors of the internal and external angles cut the base internally and externally in the same ratio.

*** (528) If the base of a triangle and the ratio of its sides be given, the points where the internal and external bisectors meet the base may be found by cutting the base internally and externally in the ratio of the sides. The solution of this problem is effected by Prop. IX., and depends only on Prop. II.

*** (529) Since the two bisectors are at right angles (83), it follows that the vertex of the triangle must be on the circumference of a circle whose diameter is the part of the base intercepted between the bisectors. Hence (389) if the base and ratio of the sides be given, this circumference is the *locus* of the vertex.

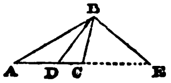
*** (530) Def.—Three magnitudes are said to be in *harmonical progression*, when the first is to the third as the difference between the first and second to the difference between the second and third.

*** (531) Def.—A right line A B is said to be cut harmonically at two points C D, when A C, A D, A B are in harmonical progression.

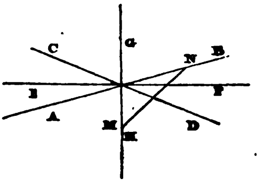


*** (532) If A C, A D, and A B be in harmonical progression, B D, B C, and B A will also be in harmonical progression. For by (hyp.) we have A C : A B = C D : D B, and by alternation A C : C D = A B : B D, and by inversion B D : B A = C D : C A, *i. e.* B D, B C, and B A are in harmonical progression.

*** (533) If the internal and external bisectors be drawn, the line A E is cut harmonically at D C, for by (327) E C : E A = C D : D A.



⁂ (534) If two indefinite right lines A B, C D intersect, and that two other indefinite right lines E F and G H bisect, the angles under these, any right line M N terminated in two of the four indefinite right lines, and intersected by the other two, will be cut harmonically. This is evident from what has been already proved (533).

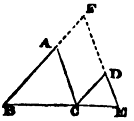


⁂ (535) It follows from this and the first proposition that the bisector of an angle of a triangle divides it into two triangles proportional to the sides which include the bisected angle.

⁂ (536) Also we may infer that perpendiculars, drawn from the point which divides the base either internally or externally in the ratio of the conterminous sides, will be equal, and *vice versâ* (521).

PROPOSITION IV. THEOREM.

(537) In equiangular triangles (B A C and C D E) the sides about the equal angles are proportional, and the sides which are opposite to the equal angles are homologous.



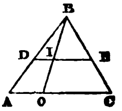
Let sides B C and C E, which are opposite to equal angles B A C and C D E, be placed so that they may form one straight line, the triangles being at the same side, and the equal angles B C A and C E D not being conterminous; since the angles A B C and B C A are together less than two right angles (XVII, Book I.), and C E D is equal to B C A, A B C and C E D are less than two right angles, and therefore the lines B A and E D must meet if produced (Ax. XII. Book I.); let them meet in F; because the angles B C A and C E D are equal (hyp.), C A is parallel to E F (XXVIII, Book I.), and because the angles A B C and D C E are equal, C D is parallel to B F (XXVIII, Book I.), therefore A F D C is a parallelogram, and the side A C equal to F D, and A F also equal to C D (XXXIV, Book I.).

In the triangle B F E the line A C is parallel to F E, therefore B A is to A F, or to C D equal to A F, as B C to C E (II); and by alternation, A B is to B C as C D to C E; and since C D is parallel to B F, B C is to C E as F D, or A C equal to F D, to D E (II); and by alternation, B C is to C A as C E to E D, therefore, since A B is to B C as D C to C E, and B C to A C as C E to E D, *ex æquali* (XXII, Book V.), A B is to A C as D C to D E, therefore the sides about the equal angles are proportional, and those which are opposite to the equal angles are homologous.

(538) It is evident that the sides opposite to equal angles are proportional, for since A B : A C = D C : D E, by alternation we have A B : D C = A C : D E.

In describing the method of placing the triangles in this demonstration, it would have been better to have said that the triangles should be so placed, that, while the sides opposite to one pair of equal angles were in the same right line, the other equal angles should be placed so as to be externally opposite to each other, and hence the parallelism of the sides opposite to these angles would be immediately perceptible. As the demonstration at present stands, the particular case of equilateral triangles is absolutely excluded. Although the proportionality of the sides in this case is evident, yet in strictness it ought either to be included in the demonstration, or if not it should be expressly and separately mentioned.

(539) If diverging lines B A, B O, B C cut parallel lines A C, D E, the corresponding segments of the parallels will be proportional. For the triangles B D I and B A O are similar, therefore B I : B O = D I : A O. In like manner B I : B O = I E : O C, ∴ D I : A O = I E : O C, or by alternation D I : I E = A O : O C.



Thus it appears that parallels not only cut diverging lines proportionally, as proved (II), but are cut proportionally by them.

(540) *In a triangle the bisector of the base drawn from the vertex bisects every parallel to the base.*

(541) *A parallel to the base of a triangle cuts off a similar triangle.*

(542) *In equiangular triangles the perpendiculars on the sides opposite equal angles are proportional to those sides.*

For these perpendiculars form with the other sides opposite equal angles equianglar right angled triangles, and (IV) are therefore proportional to the sides.

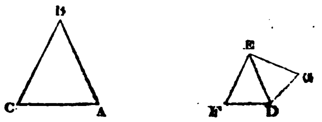
(543) *If two triangles have one angle in each equal, the perpendiculars on one pair of sides about the equal angles are as the other pair of sides.*

(544) *If two triangles have one angle in each equal, their areas are as the rectangles under the sides about those equal angles.* For they are as the rectangles under either of those sides and the perpendiculars; and the perpendiculars are as the other sides.

PROPOSITION V. THEOREM.

(545) **If two triangles (A B C, D E F) have their sides proportional (B A to A C as E D to D F and A C to C B as D F to F E) they are equiangular, and the equal angles are subtended by the homologous sides.**

At the extremities of any side D E of either triangle D E F, let the angles E D G and D E G be constructed equal to the



angles A and B at the extremities of the side A B, which is homologous to E D, and in the triangle D E G the remaining angle G is equal to the angle C in the triangle A B C (XXXII, Book I.).

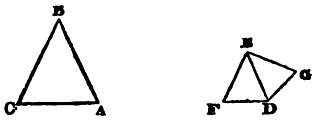
Because the triangles A B C and D E G are equiangular (const.), B A is to A C as E D to D G (IV); but B A is to A C as E D to D F (hyp.), therefore E D is to D G as E D to D F (XI, Book V.), and therefore D G and D F are equal (IX, Book V.); in the same manner it can be demonstrated that E G and E F are equal, therefore the triangle E D G is equilateral to E D F, and therefore equiangular to it (VIII, Book I.); but the triangle E D G is equiangular to B A C (const.), and therefore B A C is equiangular to E D F, and it is evident that the homologous sides subtend the equal angles.

(546) By the fourth and fifth proposition it appears that of the two requisites for similitude (Def. I.) if triangles have either, they will necessarily have the other. That figures may be similar two things are necessary: 1°. the equality of the angles: 2°. the proportionality of the sides. By the fourth, if triangles have the first requisite they will have the second, and by the fifth, if they have the second they will necessarily have the first. Triangles are in this respect unique. In all other figures, either of the requisites for similitude may subsist without the other. Thus two quadrilateral figures may have their sides proportional without having their angles equal, or *vice versâ.* A rectangle may have sides equal to those of an oblique parallelogram, and two rectangles may have sides unequal.

The property of similar triangles established in the last two propositions, and those of the right angled triangle established in the forty-seventh and forty-eighth of the first book, are by far the most important principles in the elements of geometry. On these depend the application of algebra to geometry, and they implicitly include the solution of all problems respecting rectilinear figures; for all such figures may be resolved into triangles, and a triangle may be resolved into two right angled triangles by the perpendicular.

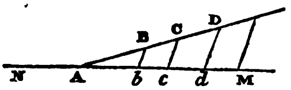
PROPOSITION VI. THEOREM.

(547) If two triangles (A B C, D E F) have one angle in each equal (A equal to D), and the sides about the equal angles proportional (B A to A C as E D to D F), the triangles are equiangular, and have those angles equal which the equal sides subtend.



At the extremities of either of the sides about the equal angles D E in the triangle D E F, let the angles E D G and G E D be constructed equal to the angles A and B at the extremities of the side A B which is homologous to E D; and in the triangle D E G the remaining angle G is equal to the remaining angle C in the triangle A B C (XXXII, Book I.).

Since the triangle A B C is equiangular to D E G (const.), B A is to A C as E D to D G (IV), but B A is to A C as E D to D F (hyp.), therefore E D is to D G as E D to D F (XI, Book V.), and therefore D G and D F are equal (IX, Book V.); the angles E D G and E D F are also equal, because each of them is equal to the angle A (const.), and the side E D is common to both; therefore the triangle E D F is equiangular to E D G (IV, Book I.); but B A C is equiangular to E D G (const.), therefore B A C is equiangular to E D F, and it is evident that the equal sides subtend the equal angles.



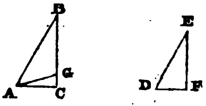
⁂ (548) From this proposition it follows, that if through any points *b*, *c*, *d*, &c. of a right line M N parallels *b* B, *c* C, *d* D, &c. be drawn and are proportional to the distances A *b*, A *c*, A *d*, &c. from any point A on that right line, their extremities B, C, D, &c. will be on the same right line passing through the point A.

For since A *b* : *b* B = A *c* : *c* C, and the angles A *b* B and A *c* C are equal, the triangles A *b* B and A *c* C are similar, ∵ the angles B A *b* and C A *c* are equal; and since the sides A *b* and A *c* coincide, and the other sides A B and A C lie at the same side of them, they must also coincide. And the same will apply to the points D, &c.

It is on this principle that the equation of a right line in analytic geometry depends.

PROPOSITION VII.

(549) If two triangles (A B C, D E F) have one angle in each equal (B equal to E), the sides about two other angles proportional (B A to A C as E D to D F), and each of the remaining angles (C and F) either less or not less than a right angle, the triangles are equiangular, and those angles are equal about which the sides are proportional.



Because the angles B A C and E D F are equal, for if it be possible let one of them B A C be greater than the other, and at the point A with the right line A B make the angle B A G equal to the less angle E D F.

In the triangles D E F, A B G the angles E and B are equal (hyp.), and E D F and B A G are also equal (const.), therefore E F D is equal to B G A (XXXII, Book I.), therefore the triangles are equiangular, and B A is to A G as E D to D F (IV); but B A is to A C as E D to D F (hyp.), therefore B A is to A G as B A is to A C (XI, Book V.), and therefore A G is equal to A C (IX, Book V.), therefore the angle A C G is equal to A G C (V, Book I.) and each of them acute (XVII, Book I.); since A G C is acute, A G B must be obtuse, and also E F D which is equal to A G B, but since A C G is acute E F D must also be acute, which is absurd.

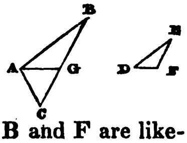
The angle B A C, therefore, is not greater than E D F; and in the same manner it can be demonstrated that E D F is not greater than B A C; they are therefore equal, and since the angles A B C and D E F are also equal (hyp.), the triangles are equiangular (XXXII, Book I.), and therefore have the sides about the equal angles proportional (IV).

The demonstration of this proposition is unnecessary prolix in Simson's edition. We would propose, however, to remodel this entire proposition as follows.

(550) *If two triangles* (A B C, D E F) *have two sides in the one proportional to two sides in the other* (B A : A C = E D : D F), *and the angles* (B, E) *opposite one pair of homologous sides* (A C, D F) *equal, the angles* (C, F) *which are opposite the other pair of homologous sides* (A B, D E) *will be either equal or supplemental.*

The angles A, D included by the proportional sides must be either equal or unequal.

1°. Let them be equal; since the angles B, E are equal (hyp.) the angles C and F must be also equal.



2°. Let them be unequal; let A be the greater, and let the construction described in the proposition be made. It follows as in the proposition that the angle A G B = F and that A G C = A C G. But A G B and A G C are supplemental, therefore A C B and F are likewise supplemental.

Hence it follows that if besides the proportionality of two sides, and the equality of the angles opposite to one pair of homologous sides, any circumstance be given which proves that the angles opposite the other pair of homologous sides are not supplemental, they must be equal, and the triangles must be similar. The circumstances which can determine this have been already mentioned in (108) *et seq.*

If either of the angles C or F be known to be right, they will be both equal and supplemental, and the triangles will be similar.

(551) The several criterions for determining the similitude of two triangles, established in this and the preceding propositions, may be enumerated as follows:

1°. The equality of the angles (IV).

2°. The proportionality of the sides (V).

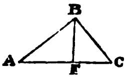
3°. The equality of two angles and the proportionality of the containing sides (VI).

4°. The proportionality of two sides in each, the equality of the angles opposite one pair of homologous sides, and any circumstance which determines either that the angles opposite the other pair of homologous sides are not supplemental or that one of them is right.

PROPOSITION VIII. THEOREM.

(552) In a right angled triangle (A B C), if a perpendicular (B F) be drawn from the right angle upon the opposite side, it divides the triangle into parts which are similar to the whole and to one another.

In the triangles A F B, A B C the angle A F B is equal to the angle A B C (hyp.), and the angle A is common to both, therefore the remaining angle A B F is equal to the remaining one C (XXXII, Book I.), and the triangles are equiangular, therefore the sides about the equal angles are proportional (IV) and the triangles are similar.



In the same manner it can be demonstrated that the triangle B F C is similar to the triangle A B C.

Since the angle A B F is equal to the angle C, and the angles A F B and B F C are also equal, the remaining angles A and F B C are equal, and the triangles A B F and B C F are equiangular, therefore the sides about the equal angles are proportional (IV), and therefore the triangles are similar.

(553) From the similitude of the whole triangle and the partial ones, we may infer that each side is a mean proportional between the hypotenuse and conterminous segment. Because A B C and A F B being similar, we have A C : A B = A B : A F (IV). And in like manner A C : B C = B C : C F.

(554) In like manner it follows that the hypotenuse is to either side as the other side is to the perpendicular, or A C : A B = B C : B F.

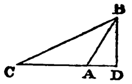
(555) Also from the similitude of the partial triangles it follows that the perpendicular is a mean proportional between the segments or A F : F B : F C.

⁂ (556) In a triangle, if the perpendicular form with the sides similar right angled triangles, the angles which it makes with the sides must be either equal or complemental.

⁂ (557) If they be equal, and therefore it bisects the vertical angle, the partial triangles must not only be similar but equal, and the whole triangle must be isosceles.

⁂ (558) If they be not equal, and the perpendicular fall within the base, they must be equal to the alternate base angles respectively, and the vertical angle will then be equal to the sum of the base angles, and will therefore be right, and the angles under the perpendicular and the sides will be complemental.

⁂ (559) If the perpendicular fall without the base, and the triangles contained by it and the sides be similar, the angles under the perpendicular and sides must be complemental, for C B D = B A D and A B D = B C D, $\because$ C B D and A B D are complemental.



⁂ (560) If the perpendicular from the vertex of a triangle on the base be a mean proportional between the segments, the right angled triangles contained by it and the sides must be similar. This easily follows by Prop. VI. Hence if in this case the perpendicular fall within the base the vertical angle is right.

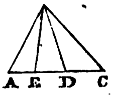
⁂ (561) If one side of a triangle be a mean proportional between the base and conterminous segment, the right angled triangle included by that segment and the perpendicular will be similar to the whole, providing that the perpendicular fall within the base; for in this case the angle included by the side and base is common to the two triangles, which are therefore similar (VI). But if the perpendicular fall without the base, the angles included by the proportional sides will not be common, but will be supplemental. The converse of (553) is therefore only true when the perpendicular falls within the base.

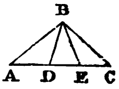
⁂ (562) If each side be a mean proportional between the base and conterminous segment, the perpendicular must fall within the base, for if it fell without it, the greater side would be greater than both the conterminous segment and the base, and therefore could not be a mean between them. Hence in this case the component triangles are similar, and the whole triangle is right angled.

⁂ (563) If the base of a triangle, the two sides, and the perpendicular be four proportionals, the triangle must be right angled, for in the whole triangle and one of the component triangles there are two sides proportional, the angles opposite to one pair of homologous sides equal, and of the angles opposite to the other pair of homologous sides one is a right angle, therefore (VII) (350) the whole triangle is similar to the partial triangle.

⁂ (564) The eighth proposition, and the consequences which we have deduced from it, are only particular cases of a more general principle.

Let A B C be any triangle, and draw B D and B E, making the angles B D A and B E C each equal to the angle A B C. The triangle D B E will then be isosceles, and the triangles B D A and B E C will be similar to each other and similar to the whole. When the





angle A B C is obtuse, the angles B D A and B E C are the external angles at the base of the isosceles, and when A B C is acute, they are internal. As the obtuse angle A B C decreases and approaches to a right angle the base D E of the isosceles triangle diminishes, and the sides B D, B E approach each other and actually coincide when the angle A B C is right. In the general proposition, this isosceles triangle D B E is what the perpendicular is when the given triangle A B C is right angled. Accordingly we find that the sides of this triangle, and the triangles under them and the sides of the given triangle, possess many of the properties already proved in the case of a right angled triangle. The student will find no difficulty in establishing the following, and in perceiving their analogy to what has been already proved:—

$AC : AB : AD,$
$AC : CB : CE,$
$AC : AB = BC : BE,$
$AD : BD : CE, (BD = BE).$

It may be an useful exercise for the student to examine whether the converses of these are true, or of what modifications they may be susceptible.

Since $AD : BD = AB : BC$, it follows that the segments A D and E C are in the duplicate ratio of the sides A B and B C.

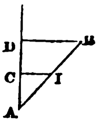
⁂ (565) Hence in a right angled triangle the segments of the hypotenuse by the perpendicular are in the duplicate ratio of the sides.

PROPOSITION IX. PROBLEM.

(566) From a given finite right line (A B) to cut off any part required.

From either extremity A of the given line draw A D, making any angle with A B; in it take any point C and make A D the same multiple of A C that A B is of the required part; join B D, and draw through C a right line C I, parallel to B D; A I is the part required.

For A I is to A B as A C to A D (II), therefore whatever multiple A D is of A C, A B is the same multiple of A I.



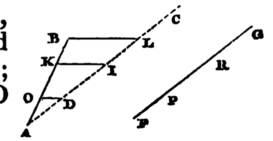
The word 'part' here means aliquot part or submultiple. This is equivalent to the problem to divide a right line into any number of equal parts.

It is evident that (X, Book I.) is a particular case of this problem.

PROPOSITION X. PROBLEM.

(567) To divide a given right line (A B) similarly to a given divided line (F G).

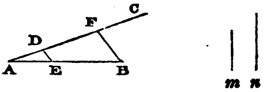
From either extremity A of the given line A B draw A C making any angle with it; take A D, D I, and I L equal to the parts of the divided line F P, P R, and R G (II, Book I.); join L B, and draw through I and D lines I K and D O parallel to L B.



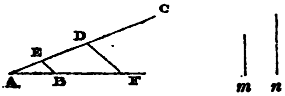
Since in the triangle B A L, the lines K I and O D are parallel to B L, B K is to K O as L I to I D (II), or as G R to R P (const.), and K O is to O A as I D to D A (II), or as R P to P F (const.), and therefore the given line A B is divided similarly to F G.

A right line is said to be cut similarly to another when its several segments are proportional to those of the other.

(568) By this proposition a line may be cut internally or externally in a given ratio. Let A B be the given line, and let the given ratio be that of $m : n$. Draw A C at any angle with A B, and take upon it A D and D F equal to m and n, and draw B F and D E parallel to B F; F will be the point of section required. Since a segment A E might be taken from the extremity B, there are evidently two points at which the line A B can be cut internally as required, and these points are equally distant from the extremities or from the middle point. If the ratio be of equality, this problem becomes the tenth of the first book. In this case the two points of section coincide, and the problem has but one solution.



If it be required to cut A B externally in the ratio $m : n$, take A D $= n$ and D E $= m$, draw B E and parallel to it draw D F. The point F will cut the line as required. For A F : B F $=$ A D : D E $= n : m$. It is evident that a point taken in the production of the line beyond A, at the same distance from A as F is from B, will also cut the line as required.



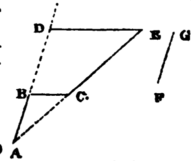
⁂ (569) It is evident that a line cannot be cut externally in a ratio of equality, since the line itself is always the difference of the segments. It is, however, sometimes considered that the point of equal external section is a point in the produced part at an infinite distance. The meaning of this will be perceived, if the lesser term of the ratio m or D E be supposed to approach to equality with n or D A. In that case, the point E continually approaching A, the line E B will continually approach to coincidence with A B, and the parallel to B E from D will continually approach to parallelism with A B, and therefore the point F will continually recede from B. When m and n are equal, the point E actually coincides with A, and the line E B with A B, and therefore the line parallel to E B through D is parallel to A B, and its intersection with A B is removed to an infinite distance.

In strictness, however, this expression of 'section by a point at an infinite distance,' should be only understood as expressing the limit of the varying construction, as the given ratio $m : n$ approaches indefinitely near to a ratio of equality.

PROPOSITION XI. PROBLEM.

(570) To find a third proportional to two given right lines (A B and F G).

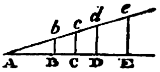
At either extremity of the given line A B draw A E making any angle with it; take A C equal to the other given line F G, and join B C; in A B produced take B D equal to F G, and through D draw D E parallel to B C; C E is the third proportional to A B and F G.



For in the triangle D A E, B C is parallel to D E, therefore A B is to B D as A C to C E (II); but B D and A C are equal to F G (const.), therefore A B is to F G as F G is to C E.

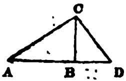
⁂ (571) If it be required to continue the progression, a repetition of the same construction will solve the problem. Let A B and B C be the given antecedent and consequent, and take A *b* = B C and draw the parallels B *b*, C *c*. Take C D = *b c*, and draw the parallel D *d*. In like manner take D E = *c d*, and draw the parallel E *e*, and so on. It is evident that

A B : B C : C D : D E, &c.

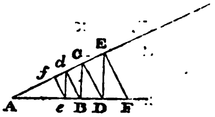


⁂ (572) There are various other constructions by which this problem may be solved. The following are very obvious.

Let A B, B C at right angles be the antecedent and consequent; join A, C, and draw C D perpendicular to A C. Then A B : B C : B D (555).



Let A B be the antecedent, and B C a line perpendicular to it. If the antecedent be less than the consequent, inflect A C on B C equal to the consequent, and produce A B and A C indefinitely beyond the points B and C. Draw C D perpendicular to A C, and we have A B : A C : A D (553), so that A D is the third proportional. If, however, the consequent be less than the antecedent, let it be A B, and let A C be the antecedent. In this case draw B *d* perpendicular to A C, and A *d* is the third proportional (553). In each case the series may be continued. From D draw D E perpendicular to A D, and from E, E F perpendicular to A E, &c. and we have (553).



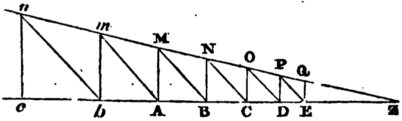
A B : A C : A D : A E : A F, &c.

Also from *d* draw *d e* perpendicular to A *d*, and from *e*, *e f* perpendicular to A E, &c. and we have

A C : A B : A *d* : A *e* : A *f*, &c.

⁂ (573) A ratio may be continued in a series, also by the following construction.

Let A B be one of the terms of the ratio, and draw the lines A M and B N perpendicular to it, and take $AM = AB$ and B N equal to the other term of the ratio, and draw M N. The ratio not being supposed to be a ratio of equality M N will not be parallel to A B, and therefore these lines will meet at Z, if produced on the side of the lesser term B N of the ratio. First let A M be the antecedent and B N the consequent. Draw the line M B, and from N draw N C parallel to M B, and from C draw C O parallel to B N. Again from O draw O D parallel to M B, and from D draw D P parallel to B N, and so on. Then we have $AB : BC : CD : DE$, &c.



For (IV) $AZ : AM = BZ : BN$,
Conv. $AZ : BZ : CZ$,
∵ $AM : BN : CO$,
∵ $AB : BC : CD$.

Hence C D is the third proportional, and by continuing the process D E will be the next term of the series, and so on. It appears therefore that if each perpendicular A M be equal to the intercept A B of the base between it and the next perpendicular, those perpendiculars will be in geometrical progression.

Next, let B N be the antecedent and A M the consequent. Draw B M, and parallel to it draw A *m* to meet Z M produced at *m*. Draw *m b* parallel to M A. In like manner draw *b n* and *n c* parallel to B M and M A, and so on. We have then

$$CB : BA : Ab : bc, \text{ \&c.}$$

This is evident from what has been already stated.

⁂ (574) *If a series of magnitudes* A, B, C, D, *be in continued proportion, their successive differences a, b, c, d, are also in continued proportion and in the same ratio. For since*

$A : B : C$,
Conv. $A : a = B : b$,
Alt. $A : B = a : b$.
In like manner we find $B : C = b : c$.
∵ $a : b : c$;

and by continuing the same process we have

$$a : b : c : d : \text{\&c.}$$

⁂ (575) *If a series in continued proportion be an increasing one, there is no limit to the increase of its terms.*

As before, let the series be $A : B : C : D$ . . . . . and let a be the excess of B above A, b the excess of C above B, and so on. Now by continuing the series there is no magnitude so great that we may not obtain a greater. Let M be any magnitude however great. Find how often the magnitude a is contained in M, and continue the proposed series through a greater number of terms. The last term will then exceed A by the sum of the series a, b, c, d, &c. continued to as many terms as the number of times that a is contained in m. But since the series $a, b, c, \ldots$ is increasing (574), each succeeding term in it is greater than a, and therefore their sum increased by A must be greater than M.

⁂ (576) *If a series in continued proportion be a decreasing one, there is no limit to the diminution of its terms.*

Let the series be $A : B : C : \ldots\ldots\ldots$ it may be continued until a term is found less than any assigned magnitude m, however small. For let l be such a magnitude that $m : l = B : A$, and let the ratio $m : l$ be continued in a series (573). Since $m > l$ (for $B > A$) this series increases, and therefore it may be continued until a term a is found greater than A (575). This being done, let the series $A : B : C : \ldots$ be continued through the same number of terms, and its last term M will be less than the assigned magnitude m. For in the two series

$$A : B : C : D \ldots\ldots : L : M,$$
$$a : b : c : d \ldots\ldots : : l : m,$$

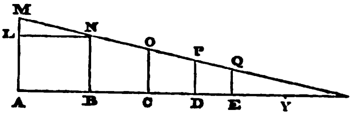
each pair of successive terms are in the same ratio, therefore *ex æquo ordinatè*

$$A : M = a : m,$$
$$\text{Alt. } A : a = M : m;$$

but A is less than a, and therefore M is less than m.

⁂ (577) *If a series of magnitudes decreasing in continued proportion be continued or imagined to be continued to an infinite number of terms, the sum of all the terms or the sum of the series will be a finite and determinate magnitude.*

Resuming the construction in (573), let the decreasing series be AB : BC : CD : DE : &c. Its sum, if the number of terms be unlimited, is AZ. It is not greater than AZ, for each perpendicular, EQ, is less than the magnitude EZ, from which it is to be taken in order to obtain the point from which the next perpendicular is to be erected. By the proportion (IV)



$$AM : AZ = EQ : EZ,$$

it follows that since $AM <^{*} AZ$, $\because EQ < EZ$.

Neither can the sum when the series is unlimited in its number of terms be less than AZ, for if it were let it be equal to AY. Now since the perpendiculars AM, BN, CO, &c. are in decreasing continued proportion, the lines AZ, BZ, CZ, &c., which are proportional to them (IV), are also in decreasing continued proportion. This series may then be continued through a determinate number of terms, so that a term may be found which is less than YZ. This being done, the sum of the corresponding perpendiculars must be greater than AY, but these are the terms of the proposed series. Hence it appears that the sum of a *limited* number of terms of the proposed series is greater than the sum AY of an *unlimited* number, a part greater than the whole, which is absurd. Therefore AZ is not greater than the sum of the series when the number of terms is increased without limit.

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* The sign < signifies less than, and > signifies 'greater than.'

If the magnitudes in the proposed series be not lines, yet lines being taken proportional to them the same conclusions may be obtained.

The sum may immediately be obtained from the first two terms A B and B C. For draw N L parallel to A Z. Then

$$ML : LN = MA : AZ.$$

But $LN = MA = AB$, therefore

$$AB - BC : AB : AZ.$$

The sum of the series is therefore a third proportional to the difference of the first and second terms and the first term.

⁂ (578) Hence, and from subsequent propositions, it follows that of the three quantities, the first and second terms and the sum of the series, if any two be given the remaining one may be found.

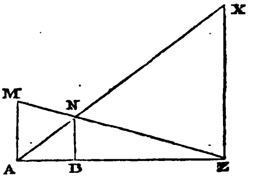
The case where the first and second terms are given has been just noticed. If the sum of the series (A Z) and the first term (A B) be given, a third proportional to them will be (A B — B C) the difference between the first two terms, which being taken from the first term leaves a remainder equal to the second.

If the sum (A Z) and the difference (A B − B C) be given, a mean proportional (XIII) between them will be the first term, from which and the difference the second may be inferred.

Let the sum (A Z) and the second term B C be given to find the first. We have (573)

$$AB : AZ = BC : BZ.$$

therefore (XVI) $AB \times BZ = AZ \times BC$, ∵ A Z is divided at B, so that the rectangle under its parts is equal to $AZ \times BC$. Let it be so divided (297), and either segment A B will be the first term of the series.

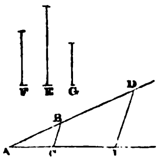


It is easy to see that whichever segment of A Z be taken as the first term, the sum of the series and the second term will be the same. For if A N be drawn and produced to meet a perpendicular through Z at X, we shall have $BZ = ZX$. For $AB : BN = AZ : ZX$. But also by what has just been proved $AB : AZ = BN : BZ$, and alt. $AB : BN = AZ : BZ$ ∵ $BZ = ZX$. Hence Z A is the sum of the series whose first term is B Z and second term B N.

Thus it appears that, when the sum of the series and second term are the data, the problem has two solutions.

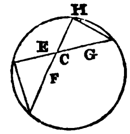
PROPOSITION XII. PROBLEM.

(579) To find a fourth proportional to three given lines (F, E, and G).



Draw two lines A D and A I making any angle in A D, take A B and B D equal to F and E, and in A I take A C equal to G, join B C, and through D draw D I parallel to B C; C I is the fourth proportional to F, E, and G.

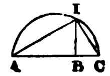
For in the triangle D A I, B C is parallel to D I; therefore A B is to B D as A C to C I (II); but the given lines F, E, and G are equal to A B, B D, and A C (const.), therefore F is to E as G is to C I.



This problem may be solved by various other constructions; *ex. gr.* Let E and G be placed in the same straight line, and from their common extremity C, in any direction, draw a line equal to F. Describe a circle through the extremities of the three lines E, F, G, and produce E to meet its circumference at H; C H is the fourth proportional sought. This easily follows from the similitude of the triangles.

Proposition XIII. Problem.

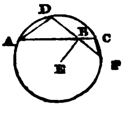
(580) To find a mean proportional between two given right lines (E and F).



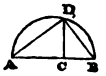
Draw any right line A C, take in it A B and B C equal to E F, and bisect A C in D; from the centre D with the radius D A describe a semicircle A I C, and through B draw B I perpendicular to A C and meeting the circumference in I: B I is the mean proportional between E and F.

Draw A I and I C. Since in the triangle A I C the angle I is right (XXXI, Book III.), and I B is perpendicular from it upon the opposite side, I B is a mean proportional between A B and B C (555), and therefore between the given lines E and F, which are equal to A B and B C (const.).

(581) There are various other constructions by which this problem may be solved.



Let A B and B C be the extremes. Describe any circle through A and C, and let E be its centre, and draw B E and D B perpendicular to B E; D B is the sought mean. For D B = B F (III, Book III.), and the triangles D B A and C B F are similar, ∵ &c.



Again, let A B and B C be the extremes, and on A B describe a semicircle. Draw C D perpendicular to A B, and draw D B, which is the mean sought.

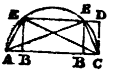
Also, it may be solved thus: let A B and B C be the extremes, and describe any segment on A B, and draw C D, making D C B equal to an angle in the segment; D B is the mean (553).

Or, if a segment be described on A C, and a tangent B D be drawn to it from B, this tangent is the mean.

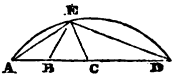
(582) By this proposition and the eleventh it appears, that of three lines in continued proportion, if any two be given, the remaining one can be found.

⁂ (583) Also, if any one and the sum of the other two be given, the other two may be found severally.

1°. Let the mean and the sum of the extremes be given. On A C, the sum of the extremes, describe a semicircle, and draw the perpendicular C D equal to the given mean, and through D draw D E parallel to A C, meeting the semicircle in E, and draw E B perpendicular to A B. Since the triangle A E B is right-angled, B E is a mean between A B and B C (555), which are therefore the extremes. It is evident that each of the points E will give the same extremes.

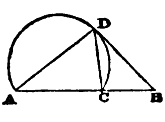


2°. If one extreme and the sum of the mean and the other extreme be given; on A D, the sum of the three, describe the segment of a circle containing an angle equal to the external angle of an equilateral triangle, and take A B equal to the given extreme; draw B E, making A B E equal to an angle in the segment, and E C, making E C D equal to the same angle. B E C is an equilateral triangle, and B E or B C is a mean proportional between A B and C D, so that we have A B : B C : C D. Hence B C and C D are the mean and the other extreme.



⁂ (584) Also, if any one of three lines in continued proportion and the difference of the other two be given, the other two may be found.

1°. Let the mean and the difference of the extremes be given. On the given difference A C describe any segment of a circle A D C, and inflect a tangent between the circle and produced chord A C B, so that B D shall be equal to the given mean (354). In this case it is clear that B D is a mean between A B and B C, from the similitude of the triangles A D B and B D C.



2°. Let one extreme and the difference between the mean and the other extreme be given. In the solution of this case we shall anticipate the sixteenth proposition. Since in continued proportion the differences of the successive terms are as the terms themselves (574), it follows that the rectangle under one extreme and the difference of the mean and the other is equal to the rectangle under the mean and the difference between it and the given extreme (XVI). Since, then, the area of this rectangle and the sum or difference of its sides are given, the sides themselves may be found.

⁂ (585) By the thirteenth proposition, 3, 7, 15, &c. means may be found between two given lines. For having found one mean, means may be found between it and each extreme, and thus we shall have three means. Inserting again means between every pair of successive terms of the series thus found, we shall have 7 means, and in the same manner 15 means may be obtained, and so on. Any number of means which is one less than a power of 2 may thus be found, the powers of 2 being 2, 4, 8, 16, 32, &c.

(586) The problem to determine *two* mean proportionals between two given lines, which produced so much discussion among the ancient mathematicians, has never been solved geometrically. The circumstance which has rendered the solution of this problem so desirable is, that upon it depends the solution of a most important problem in solid geometry, *viz.* 'to construct a solid of a given species and given magnitude,' or 'to construct a solid similar to a given one, and bearing to it a given ratio.' It is an established principle of solid geometry that similar solids are in the triplicate ratio of their homologous rectilinear edges. Hence to find a solid similar to a given one bearing a given ratio to it, it is only necessary to find a line which shall bear the given ratio to one of its edges; and if two mean proportionals be found between this line and the edge of the given solid, the similar solid, with the first of these means as an edge, will be that which is required for the construction of the solid.

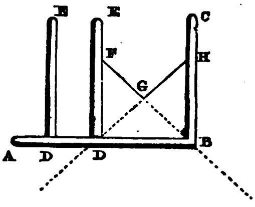
The problem to vary the scale of solids in any required proportion is so very obvious a geometrical inquiry, that it most probably first attracted the attention of the ancient geometers to the investigation of two mean proportionals. Nevertheless, an ancient author ascribes the origin of this problem to the following occurrence. A plague happening to rage in Greece, deputies were sent to consult the Delphic Oracle as to the means of assuaging it. The Divinity answered that if his altar, which was of a cubical form, were doubled, its shape being retained, he would remove the evil. The ignorant deputies accordingly doubled its edges, by which its capacity or solid dimensions were in fact increased eight times. The plague still raging, the deputation returned and received the same answer. The geometers were now consulted, and the problem was brought to Plato, the first mathematician of the age. Plato immediately perceived its difficulty and declined it, referring the solution of it (according to Valerius Maximus) to Euclid. And since that time the matter has remained undetermined. The problem has been hence called 'the Delian Problem.'

This tale, however, bears strong marks of fiction. Among others is the circumstance of an anachronism in referring Euclid to the time of Plato, he having flourished half a century after him. It is much more probable that the whole tale is a fabrication of an early writer or mathematician of minor note, invented to give an adventitious importance to the problem.

This question was, however, raised at a very early period, and received the name of 'the duplication of the cube' from the fable we have related. *Hippocrates* of *Chios* first reduced it to the determination of two mean proportionals. Failing in the geometrical solution of the problem, various mechanical means have been from time to time suggested, some of which we shall now mention. By mechanical means we would be understood to mean some instrument different from the rule or compass, which are the only ones allowed in geometry.

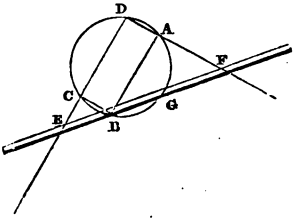
⁂ (587) One of the earliest solutions of this problem is that of Plato.

Let A B C be two straight rulers fixed at a right angle, and let D E be a ruler sliding along A B, but always perpendicular to it, so as to be capable of being successively moved into the positions D E. Let the given extremes F G and G H be placed at right angles and produced indefinitely through the vertex G of the right angle. Let the instrument now be so placed that while the production of one extreme passes through the right angle B, the sliding ruler D E may be so moved that when it passes through the extremity F of that extreme it will also pass through the point D where the production of the other extreme H G meets the ruler B A, and at the same time let the other rulers B C be made to pass through H. This being done, the intercepts G B and G D are the two means. For since H B D is a right angle (555), H G : G B : G D, and since B D F is a right angle, B G : G D : G F.



*** (588) Another ancient geometer (*Philo* of *Byzantium*) has imagined the following solution.

Let the extremes A B, B C be placed at right angles, and the rectangle completed, and a circle described round it. Produce D A and D C indefinitely through A and C. Let a *graduated straight* ruler be made to revolve on the point B, extending on both sides of that point, and let it be adjusted in such a position that B F shall be equal to G E. Then C E and A F will be the required means.



For by similar triangles we have the proportions

$$A B : A F = E D : D F,$$
$$C E : C B = E D : D F.$$

Also since $G E = B F$, $\therefore$ $B E \times G E = G F \times B F$. But (XXXVI, Book III.) $B E \times G E = D E \times C E$, also $G F \times B F = D F \times F A$, $\therefore$ $D E \times C E = D F \times F A$.

Since these rectangles are equal, their sides (XVI) are reciprocally proportional, therefore

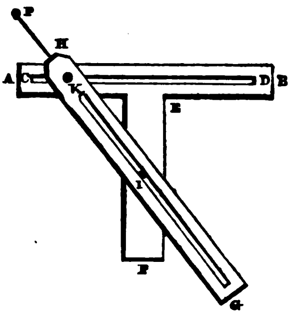
$$A F : C E = E D : D F.$$

From this and the former propositions we find

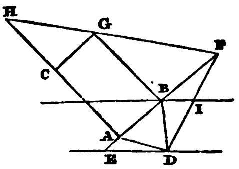
$$A B : A F : C E : C B.$$

*** (589) *Nicomedes*, a Greek geometer, who lived about two centuries before the Christian era, found that the determination of two mean proportionals depends on the solution of the problem, 'To draw a right line passing through a given point and intersecting the sides of a given angle so that the part of it intercepted shall have a given magnitude.'

To solve this problem he invented an instrument called the *Trammel of Nicomedes.*



Let A B be a flat ruler in which there is a groove C D. Attached to the middle E of this is another flat ruler E F, perpendicular to it in which, at I, is a fixed pin, which is inserted in the groove of a third ruler G H, in which there is also a fixed pin at K, which is inserted in the groove C D. The instrument being thus constructed, let a stem H P, of a length equal to that part of the line which is proposed to be intercepted by the sides of the angle, be attached to it. This done, let the fixed pin I be placed upon the given point, and the groove C D on one side of the given angle, and let the ruler H G be moved so that the pin K will move over one side of the angle, and let it be so moved until the point P shall come upon the other side of the angle. The required line will then be evidently that which joins the points P and I.



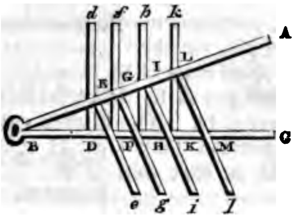
To apply this to the determination of two mean proportionals:—Let a and d be the extremes, and let a rectangle be constructed whose sides are equal to the extremes $AB = a$, $AC = d$. On A B construct an isosceles triangle B D A, the side of which B D is equal to half of A C. Produce B A so that $AE = BA$, and connect D and E, and through B draw B I parallel to D E. Through B produce A B, and through D draw D F by the trammel, so that $IF = BD$, and draw F G intersecting A C produced in H. Then $BF = b$, and $CH = c$, b and c being the sought means.

For since B I and D E are parallel, $DI : IF :: EB : BF$, or $DI : \frac{1}{2}d :: 2a : BF$. But also on account of the similar triangles, $HC : a :: d : BF$, $\because HC = DI$ Since B D A is isosceles, the square of D F is equal to the rectangle under A F and F B, together with the square of B D. But also the square of D F is equal the square of the sum of H C, and half of A C, or to the rectangle under A H and H C, together with the square of half of A C. Taking away this last from both, it follows that the rectangle under A H and H C is equal to the rectangle under A F and F B. By this and the similar triangles we have the proportions

$$AH : AF :: AC : BF,$$
$$AH : AF :: BF : HC,$$
$$AH : AF :: HC : AB,$$
$$\because AC \cdot BF : HC : AB,$$
$$\text{or } a : b : c : d.$$

⁂ (590) There are various other mechanical solutions for this celebrated problem. We shall, however, only mention the contrivance of *Descartes*, by which any number of means may be found between two given extremes.

Let A B C be two rulers, united at their extremities B by a pivot on which they turn. In each of these rulers is a groove in which several rulers D *d*, E *e*, F *f*, &c. move so as to be always perpendicular to the grooved rulers respectively, and so that the perpendicular D *d* nearest to B, upon opening the rulers, pushes forward the ruler E *e*, and E *e* pushes forward F *f*, and so on.



Now if two means be required, let the first ruler D *d* be moved from B until B D is equal to the lesser extreme, and let B C be closed upon B A and all the perpendicular rulers moved up to D. Let the ruler D *d* be screwed to the position in which it is placed, and then let the rulers C B A be opened. The ruler D *d* will push E *e* from B, and E *e* will in like manner push F *f*, and so on. Let the rulers be opened until B G be equal to the greater extreme. Then B F and B E are the two means, as is evident from (553).

If three means be required, the rulers are to be opened until B H is equal to the greater extreme, and then B E, B F, and B G are the means.

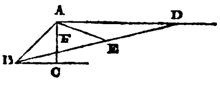
If four be required, B I is to be equal to the greater extreme, and so on.

In general, if an even number of means be required, the extremes will be on different rulers, and if an odd number be required, they will be on the same ruler.

⁂ (591) We have before alluded to another problem which has baffled the skill of geometers, *scil.* 'the trisection of an angle.' Indeed the modern analysis shows that the solutions of both these problems depend on the same principles, and that neither of them can be solved by the circle and right line, but require the aid of an higher geometry. It is scarcely necessary to observe that the investigations respecting their geometrical solution are purely speculative, since they can be solved practically and analytically with any degree of accuracy.

They were early discovered to depend on the same principle. Nicomedes showed that both could be solved by the *trammel*.

Let A B C be the angle to be trisected. From A draw A C perpendicular to B C, and from A draw A D parallel to B C. Inflect (by the *trammel*) B D so that F D shall be equal to twice B A. Then the angle F B C is one third of A B C, and if A B F be bisected the angle A B C will be trisected.



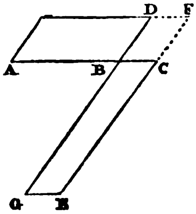
For bisect F D at E, and join A E. Since F A D is a right angle A E = E D = F E, therefore the angles E A D and A D E are equal, and A E B is equal to twice A D B; but A D B = D B C (XXIX, Book I.), and therefore A E B is twice D B C; but A E = A B, ∴ A E B = A B E, ∴ A B E is equal to double D B C, and ∴ F B C is one third of A B C.

Proposition XIV. Theorem.

(592) 1°. Equal parallelograms (A D and G C), which have one angle in each equal, have the sides about the equal angles reciprocally proportional (A B to B C as B G to B D).

2°. And parallelograms which have one angle in each equal, and the sides about them reciprocally proportional, are equal.

Part 1°.—Let the sides A B and B C be so placed that they may make one right line, and that the equal angles may be vertically opposite; since A B D and D B C are equal to two right angles (XIII, Book I.) and G B C is equal to A B D (hyp.), G B C and D B C are equal to two right angles, and therefore G B and D B form one right line (XIV, Book I.). Complete the parallelogram D C.



Since the parallelograms A D and G C are equal (hyp.), A D is to D C as G C to D C (VII, Book V.); but A D is to D C as A B to B C (I), and G C is to D C as G B to B D (I), therefore A B is to B C as G B to B D.

Part 2°.—Let the same construction remain; A D is to D C as A B to B C, and G C is to D C as G B to B D; but A B is to B C as G B to B D (hyp.), therefore A D is to D C as G C to D C (I), and therefore the parallelogram A D is equal to the parallelogram G C (IX, Book V.).

(593) The sides of two figures are said to be *reciprocally* proportional when the extremes of the proportion are sides of one figure, and the means are sides of the other.

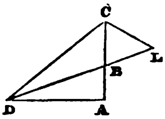
On the other hand, they are said to be directly proportional when two sides of each figure are a mean and an extreme.

Of the three properties contemplated in this proposition, *scil.* 1°. the equality of the angles; 2°. the reciprocity of the sides; 3°. the equality of the areas: if any two of them be given, the third may always be inferred. Of the three cases to which this inquiry resolves itself two are determined in the proposition. The third is, that 'if two parallelograms have equal areas, and their sides reciprocally proportional, they will be equiangular.' For the proof of this, see observations on Prop XVI.

PROPOSITION XV. THEOREM.

(594) 1°. Equal triangles, which have one angle in each equal (A B D equal to C B L), have the sides about the equal angles reciprocally proportional (A B to B C as L B to B D).

2°. And two triangles (A B D and C B L) which have one angle equal, and the sides about the equal angles reciprocally proportional, are equal.



Part 1°.—Let two of the sides A B and B C about the equal angles be so placed that they may form one right line, and that the equal angles may be vertically opposite; then since A B D and D B C are equal to two right angles (XIII, Book I.), and L B C is equal to A B D (hyp.), D B C and L B C are equal to two right angles, therefore D B and B L form one right line (XIV, Book I.): join D C.

Since the triangles A B D and L B C are equal, A B D is to D B C as L B C is to the same D B C (VII, Book V.); but A B D is to D B C as A B to B C (I), and L B C is to D B C as L B to B D (I), therefore A B is to B C as L B is to B D.

Part 2°.—Let the same construction remain, and A B D is to D B C as A B to B C, and L B C is to D B C as L B to D B (I); but A B is to B C as L B to D B (hyp.); therefore A B D is to D B C as L B C to D B C (XI, Book V.), and therefore A B D is equal to L B C (IX, Book V.).

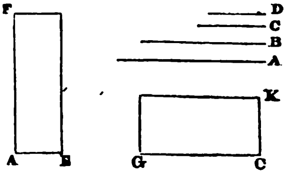
This proposition might have been inferred from the last, since the triangles are the halves of equiangular parallelograms. This consideration also shows that the same property extends to the case of triangles in which the angles included by the reciprocal sides are supplemental.

PROPOSITION XVI. THEOREM.

(595) If four right lines be proportional (A to B as C to D), the rectangle under the extremes (A and D) is equal to the rectangle under the means (B and C).

And if the rectangle under the extremes be equal to the rectangle under the means, the right lines are proportional.

Part 1°.—Draw A E and G C equal to D and C, and erect A F and C K perpendicular to them and equal to A and B; complete the rectangles E F and G K.



Because in the parallelograms E F and G K the angles A and C are equal, and the sides about them reciprocally proportional (hyp.), E F is equal to G K (XIV).

Part 2°.—Let the same construction remain: because the parallelograms E F and G K are equal (hyp.) and the angles A and C are equal, A F is to C K as G C to A E, and therefore A is to B as C to D (XIV).

This proposition (of which the succeeding is a particular case) is one of the most important in the Elements, and in its fertility equals the celebrated forty-seventh of the first book. The following principles, which are very generally useful in geometry, give this proposition as a necessary consequence.

(596) Def.—*Two ratios are said to be reciprocals when the antecedent is to the consequent in one as the consequent to the antecedent in the other.*

(597) *A ratio compounded of reciprocal ratios is a ratio of equality.*

For let $A : B$ and $a : b$ be the reciprocal ratios. Since $A : B = b : a$,

$$\therefore \left.\begin{matrix}A : B\\ a : b\end{matrix}\right\} = \left\{\begin{matrix}A : B\\ B : A\end{matrix}\right\} = A : A,$$

which is a ratio of equality.

(By $\left.\begin{matrix}A : B\\ a : b\end{matrix}\right\}$ is meant a ratio compounded of $A : B$ and $a : b$.)

(598) *If a ratio of equality be compounded of two ratios they must be reciprocals.*

For if $A : A$ be compounded of two ratios, one of which is $a : b$, let the other be $c : d$, and let $c : d = b : x$, then

$$\left.\begin{matrix}a : b\\ c : d\end{matrix}\right\} = \left\{\begin{matrix}a : b\\ b : x\end{matrix}\right\} = a : x.$$

But (hyp.) $a : x$ is a ratio of equality, $\therefore x = a$. Hence $b : x$ is the reciprocal of $a : b$, and $\therefore c : d$ is the reciprocal of $a : b$.

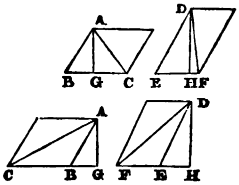
(599) By (519) it was proved that triangles and parallelograms are in a ratio compounded of their bases and altitudes. When they are equal, therefore, their bases and altitudes are reciprocally proportional (598), and if their bases and altitudes be reciprocally proportional they will be equal (597).

Hence the sides of equal rectangles are four proportionals, the means being the sides of one and the extremes of the other, and *vice versâ* the rectangle under the means is equal to the rectangle under the extremes. Thus the sixteenth proposition follows immediately from the first.

(600) It is evident that the same is true of any equiangular parallelograms, or of triangles having one angle equal or supplemental.

(601) Also if the sides of parallelograms or triangles be reciprocally proportional and their areas equal, the angles contained by the reciprocal sides will be either equal or supplemental.

Let A C B and D E F be parallelograms or triangles, and let $AB : DE = EF : BC$. Let the perpendiculars A G and D H be drawn. By (599) $AG : DH = EF : BC$, $\because$ $AB : DE = AG : DH$, and by alt. $AB : AG = DE : DH$. Hence (550) the triangles A B G and D E H are similar. If then the perpendiculars fall within both bases B C and E F, the angles A B C and D E F are the angles included by the reciprocal sides. If they fall without both bases, these angles are the supplements of the angles included by the reciprocal sides, and if one fall within and the other without, one of these angles is that included by the reciprocal sides, and the other is the supplement of that included by the reciprocal sides in the other figure. Hence in all cases the angles included by the reciprocal sides must be either equal or supplemental.



If the figures be parallelograms, they must be equiangular.

PROPOSITION XVII. THEOREM.

(602) 1°. If three right lines be proportional (A to B as B to C) the rectangle under the extremes is equal to the square of the mean.

2°. And if the rectangle under the extremes be equal to the square of the mean, the three right lines are proportional.

Part 1°.—Assume a line D equal to B, and A is to B as D to C (hyp.), therefore the rectangle under A and C is equal to the rectangle under B and D (XVI), and therefore equal to the square of B.

Part 2°.—Assume a line D equal to B; the rectangle under A and C is equal to the rectangle under D and B, therefore A is to B as D to C (XVI), and therefore A is to B as B to C.

This proposition is only that particular case of the last in which the means are equal. We shall subjoin some of the most remarkable consequences deducible from these and the preceding propositions

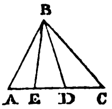
· By combining the inferences made from (VIII) with these, the following properties may be immediately deduced.

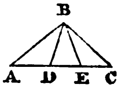
In a triangle let the sides be called A and B, the base H, and let the perpendicular on H be P, and the segments of H conterminous with A and B respectively a and b.

(603) If the angle opposite H be right the square of $P = a \times b$. Also the square of $A = a \times H$, and the square of $B = b \times H$, and $A \times B = H \times P$.

It also follows that in any triangle if the square of $P = a \times b$, the angles under P and the sides are complemental, in which case if P fall within H, the angle opposite to H is right. Also, if P fall within H, and the square of $A = H \times a$, the angle opposite H is right. Also, if $H \times P = A \times B$ the angle opposite H is right.

If from the vertex B of a triangle A B C lines B E, B D be drawn, making the angles B D A and B E D equal to the angle A B C, the square of $BD = AD \times EC$, the square of $AB = CA \times AD$, and the square of $BC = AC \times CE$; and *vice versâ*, if these equalities subsist the lines B E and B D are inclined to the base at angles equal to A B C.





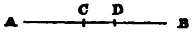
The sum of the squares of the sides A B and B C is equal to the sum of the rectangles $AC \times AD$ and $AC \times CE$. But since these rectangles have a common side A C, they are together equal to the rectangle under A C, and the sum of A D and C E (I, Book II.). If the angle A B C be obtuse, the square of A C exceeds this rectangle by the rectangle $AC \times DE$, and if A B C be acute, this rectangle exceeds the square of A C by the rectangle $AC \times DE$. Hence it follows that in every case the difference between the sum of the squares of the sides and the square of the base is the rectangle under the base A C and the base D E of the isosceles triangle D B E.

When A B C is a right angle D E vanishes, and the result becomes the forty-seventh, Book I.

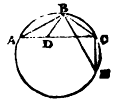
It thus appears that the forty-seventh proposition of the first book can be deduced as a consequence from the property of similar triangles established in (IV) and (V), thus the fundamental propositions of geometry are reduced to that single property.

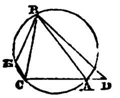
The student has already seen many instances of propositions already established reappearing in the consequences deduced from subsequent principles. One of the most striking beauties of geometry, and at the same time the most convincing proof of the certitude of its reasonings, is this constant verification of its own processes. For were it otherwise, were there the slightest want of exactitude in the results, there would be an inevitable discordance and contradiction in these consequences drawn from different sources, and they would not converge, as they do, always to the same point. There are many ways of error, but only one of truth.

If a line A B be cut harmonically at C, D, the rectangle A B × C D under the whole line and middle part is equal to the rectangle A C × D B under the extreme parts, for (530, 531) A B : B D = A C : C D.



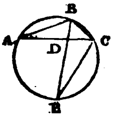
⁂ (604) *If a circle be circumscribed about a triangle* A B C, *and from the vertex* B *two lines be drawn making equal angles* D B A *and* E B C *with the sides, and one* B D *be drawn to meet the base* A C *or its production, and the other* B E *to meet the circle; the rectangle* B D × B E *under these lines is equal to the rectangle under the sides* A B × B C.





Draw C E. The angles A B D and C B E are equal (hyp.); also the angles C E B and B A D (XXI, XXII, Book III.). Hence the triangles B A D and C B E are similar; ∵ B A : B D = B E : B C (IV), ∵ B A × B C = B D × B E (XVI).

⁂ (605) If B D and B E coincide they will bisect the angle A B D, and E B × B D = A B × B C. But E B × B D = E B × B C together with the square of B D. But A B × B C = A D × D C (XXXV, Book III). Hence A B × B C = A D × D C together with the square of B D. *Hence if a line* B D *be drawn bisecting the vertical angle of a triangle the rectangle under the sides is equal to the square of that line, together with the rectangle under the segments of the base.*



⁂ (606) *If a line* B D *be drawn to the base of a triangle, so that its square together with the rectangle* A D × D C *under the segments shall be equal to the rectangle* A B × B C *under the sides, that line* B D *will bisect the angle* A B C, *except when the sides* A B *and* B C *are equal, in which case every line drawn to the base* (253) *will have the proposed property.*

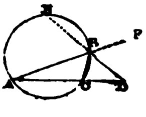
For let the circle be circumscribed and C E be drawn. Then A D × D C = B D × D E; add to both the square of B D, and A D × D C together with the square of B D, or (hyp.) the rectangle A B × B C is equal to B D × D E, together with the square of B D or (III, Book II.) B E × B D. Since B E × B D = A B × B C, we have (XVI) A B : B D = E B : B C; and since the angles B A D and B E D are equal, the angles B C E and B D A are either equal or supplemental (550).

1°. Let them be equal. The angles A B D and E B C are therefore also equal, and B D bisects A B C.

2°. Let them be supplemental. The arc B A E together with the arcs B A and C E is equal to the whole circumference (377).

Hence the arcs B A and B C are equal, and therefore their chords are equal.

⁂ (607) If B E and B D lie in the same straight line, B D will bisect the external angle F B C of the triangle. For E B A = F B D. In this case, if the square of B D be added to the rectangle E B × B D, the sum will be equal to the rectangle E D × D B, which is equal to the rectangle A D × D C (XXXVI, Book III). Hence the rectangle A D × D C = the rectangle A B × B C together with the square of B D. Hence if a line be drawn bisecting the external angle of a triangle, the rectangle under the sides together with the square of that line is equal to the rectangle (A D × D C) under the segments of the base.

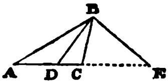


*** (608) *If a line* B D *be drawn from the vertex of a triangle to the produced base, so that the square of this line* B D *together with the rectangle under the sides* A B × B C *be equal to the rectangle under the segments of the base* A D × D C, *the line* B D *will bisect the external angle of the triangle except when the sides* A B *and* B C *are equal, in which case every line drawn to the produced base has this property.*

This may be proved nearly in the same manner as (606).

*** (609) From (605, 607) it follows, that if the bisectors B D, B E of the internal and external angles be drawn, *the rectangle under the external segments of the base exceeds the rectangle under the sides by the square of the external bisector, and the rectangle under the sides exceeds the rectangle under the internal segments of the base by the square of the internal bisector.*

Hence if the two bisectors be equal, the three rectangles A E × E C, A B × B C, A D × D C are in *arithmetical progression.* That this may take place it is necessary that the angles B D E and B E D should be equal, and therefore each half a right angle (83).



Therefore the difference of the angles B D C and B D A is a right angle, and therefore also the difference of the angles B C A and B A C is a right angle.

Hence *when the difference of the base angles is a right angle the three rectangles are in arithmetical progression.*

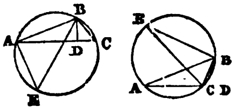
*** (610) In order that three similar figures be in continued proportion, it is necessary (XXII) that their homologous sides be in continued proportion. Hence if the rectangles A E × E C, A B × B C, and A D × D C be in continued proportion, their sides E C, B C, and D C must be in continued proportion. But since D B E is a right angle, if B C be a mean proportional between D C and C E it must be perpendicular to D E (560), ∵ B C A is a right angle. *Hence, if the three rectangles be in continued proportion or geometrical progression, one of the base angles must be right.*

*** (611) If these three rectangles be in *harmonical progression*, the first must be to the third as the difference between the first and second to the difference between the second and third. Hence A E × C E : A D × C D as the square of B E is to the square of B D. But since the rectangles are similar figures, and also the squares, we have (XXII) C E : C D = B E : B D; ∵ (III) the angles E B C and C B D are equal, and each is therefore half a right angle. But also C B D and D B A are equal, and ∵ C B A is a right angle.

Hence *the three rectangles are in harmonical progression when the vertical angle is right.*

⁂ (612) It is evident that in every case the rectangle C E × A E exceeds A D × D C by the square of D E.

⁂ (613) If the line B D drawn to the base be the perpendicular, the line B E will be a diameter. For the angle B A E is equal to B D A, and therefore is right, and B E is (XXXI, Book III.) a diameter. Hence the rectangle under the sides A B × B C of a triangle is equal to the rectangle under the altitude B D, and the diameter D E of the circumscribed circle.

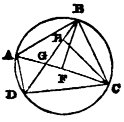


⁂ (614) It is evident that the converse of this is true; *scil.*: *That if a line be drawn from the vertex to the base such that the rectangle under it and the diameter is equal to the rectangle under the sides, that line will be the perpendicular.* Also, *if on the perpendicular as base a rectangle be constructed equal to the rectangle under the sides, the altitude of that rectangle will be equal to the diameter of the circumscribed circle.*

⁂ (615) Hence of the four lines, the two sides, the altitude and the diameter of the circumscribed circle, if any three be given, the remaining one can be found. Or if any two and the sum or difference of the other two be given, the other two can be separately found.

⁂ (616) The rectangles under two sides of any triangles inscribed in the same or equal circles, are as the perpendiculars on the third sides, for these rectangles are equal to the rectangles under the perpendiculars and equal diameters which (I) are as the perpendiculars.

⁂ (617) Hence if a quadrilateral A B C D be inscribed in a circle, the rectangles A B × B C, B C × C D under conterminous sides are as the perpendiculars B F and C E on the diagonals.



⁂ (618) From this it is easy to infer that the rectangle under the segments of each pair of conterminous sides is proportional to the conterminous segment of the diagonal. For the right-angled triangles B G F and C G E are similar, and therefore $BG : CG = BF : CE$. By applying the same principle to each pair of conterminous sides we obtain the following proportions:

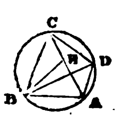
$$\begin{gathered}AB \times BC : BC \times CD = BG : CG,\\ BC \times CD : CD \times CA = CG : DG,\\ CD \times CA : DA \times AB = DG : AG,\\ DA \times AB : AB \times BC = AG : BG,\\ AB \times BC : AD \times DC = BG : GD,\\ BC \times CD : BA \times AD = CG : GA.\end{gathered}$$

By the last two proportions combined we have $AB \times BC + AD \times DC : BC \times CD + BA \times AD = BD : AC$, that is, ‘the sums of the rectangles under the pairs of sides terminated in each diagonal are as the respective diagonals.’

⁂ (619) The converse of these properties may be easily established; *scil.*: *If the rectangles under any two pairs of conterminous sides of a quadrilateral be proportional to the conterminous segments of the diagonals, the quadrilateral may be circumscribed by a circle.*

For the segments of the diagonals are proportional to the perpendiculars from the common extremities of the sides on the other diagonals. Thus B G : C G = B F : C E. Hence the rectangles are as these perpendiculars, and therefore the diameters of the circles circumscribing the triangles A B C and B C D are equal, and therefore they must be the same circle. This is evident from the principle that but one circle of a given diameter can pass through two given points.

⁂ (620) *The sum of the rectangles under the opposite sides of a quadrilateral inscribed in a circle is equal to the rectangle under the diagonals.*

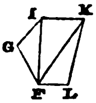


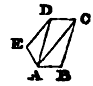
If B D do not bisect the angle A B C draw B E, making the angle C B E equal to A B D. The triangle A B D is then similar to B E C, and A B E to D B C (604). Hence A D : D B = C E : C B,
A B : A E = B D : D C.

Hence (XVI) A D × C B = B D × C E, and A B × D C = B D × A E, ∵ A D × C B + A B × D C = B D × C E + B D × A E. But the rectangles B D × C E and B D × A E under B D and the parts of A C are together equal to the rectangle under B D and A C (I, Book II.).

Proposition XVIII. Problem.

(621) On a given right line (A B) to construct a rectilinear figure similar to a given one (F G I K I) and similarly placed.





Draw F I and F K; make at the extremities of the line A B the angles B A C and A B C equal to L F K and F L K; let the lines A C and B C meet in C, and the angle B C A is equal to L K F; in the same manner construct upon A C a triangle equiangular with F K I, and so on.

The angles A B C and F L K are equal (const.); B C D and L K I are also equal, because B C A is equal to L K F and A C D to F K I (const.); and in the same manner it can be proved that the angles in the figure A E D C B are severally equal to the angles in the figure F G I K L, therefore the figures A E D C B and F G I K L are equiangular; but since the triangles A B C and F L K are equiangular (const.), A B is to B C as F L to L K (IV), and also B C to C A as L K to K F (IV); also A C D and F K I are equiangular, therefore C A is to C D as K F to K I (IV), and therefore *ex æquali* B C is to C D as L K to K I; and in the same manner it can be proved that the sides about the other equal angles are proportional, and since the figures A E D C B and F G I K L are equiangular, they are similar.

(622) The figure A E D C B is said to be placed on A B similarly to F G I K L on F L, when A B and F L are homologous sides in the two figures

If two or more sides in two polygons be equal, those only are considered homologous which are placed between angles which are equal each to each.

As many figures of the same species with different areas can be constructed on the same right line as a figure of the proposed species has sides of different lengths.

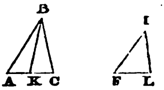
(623) Def.—A figure is said to be given in species when its several angles and the ratios of the sides about them are given.

(624) Def.—A figure is said to be given in magnitude when its area, or any figure equal to it in area, is given.

Proposition XIX. Theorem.

(625) Similar triangles (A B C, F I L) are to each other in the duplicate ratio of their homologous sides.

Take a third proportional K C to the homologous sides A C and F L, and join B K.



Since A C is to C B as F L to L I (hyp.), *alt.* A C is to F L as C B to L I; but A C is to F L as F L to C K (const.), therefore C B is to L I as F L to C K, and the angle C is equal to the angle L (hyp,), therefore the triangle K B C is equal to F I L (XV), and A B C has to both the same ratio; but A B C is to K B C as A C to K C (I), therefore A B C is to F I L as A C to K C, or in the duplicate ratio of A C to F L (369).

(626) Cor.—From this it is manifest, that if three straight lines be proportionals, as the first is to the third so is any triangle upon the first to a similar and similarly posited triangle upon the second.

In the construction for the demonstration of this proposition when a third proportional to two homologous sides has been found, a part equal to it is to be taken upon whichever side was taken as antecedent of the ratio in finding the third proportional; for otherwise the sides of the constructed triangle B C K would not be *reciprocally* proportional to those of the consequent triangle. If in taking the third proportional the lesser of the two homologous sides be taken as antecedent, the third proportional K C will be greater than the antecedent A C; in which case it will be necessary to produce A C through A, and from the produced line to take C K equal to the third proportional. It is worth notice that C K the third proportional may be taken on the production of A C through C. In this case the angles

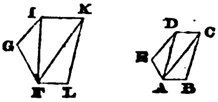
B C K and I L F will be supplemental, and the sides about them reciprocally proportional.

This proposition might be easily inferred as a particular case of the general principle deducible from the first proposition of this book, that triangles are in the ratio compounded of their bases and altitudes (519). When they are similar their altitudes are as their bases, and this compound ratio is therefore the duplicate ratio of their bases (374).

PROPOSITION XX. THEOREM.

(627) Similar polygons may be divided into similar triangles equal in number and proportional to the polygons; and the polygons are to each other in the duplicate ratio of their homologous sides.

Part 1°.—For the angles G and E are equal; and the sides about them proportional (hyp.), therefore the triangles F G I and A E D are similar (VI); since the angles G I F and E D A are equal, and also the angles G I K and E D C (hyp.), the remainders F I K and A D C are equal; and since F I is to I G as A D to D E, and I G to I K as D E to D C (hyp.), *ex æquali* F I is to I K as A D to D C, and therefore as the angles contained by them are equal, the triangle F I K is similar to A D C (VI); and in the same manner it can be proved that all the other triangles are similar.



Part 2°.—As the triangle F G I is similar to A E D, F G I is to A E D in the duplicate ratio of F I to A D (XIX), also F I K is to A D C in the duplicate ratio of F I to A D, therefore F G I is to A E D as F I K to A D C; and in the same manner it can be proved that F I K is to A D C as F K L to A C B, therefore as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents (XII, Book V.), or the polygon F G I K L to the polygon A E D C B.

Part 3°.—As the polygon F G I K L is to the polygon A E D C B as the triangle F G I to the triangle A E D, and F G I is to A E D in the duplicate ratio of the side F G to A E, F G I K L is to A E D C B in the duplicate ratio of F G to A E.

(628) COR. 1.—In like manner it may be proved that similar four-sided figures, or of any number of sides, are one to another in the duplicate ratio of their homologous sides: and it has already been proved in triangles: therefore, universally, similar

rectilinear figures are to one another in the duplicate ratio of their homologous sides.

(629) Cor. 2.—And if to A B, F G, two of the homologous sides, a third proportional M be taken, A B has to M the duplicate ratio of that which A B has to F G: but the polygon upon A B has to the polygon upon F G likewise the duplicate ratio of that which A B has to F G, therefore as A B is to M so is the figure upon A B to the figure upon F G: which was also proved in triangles: therefore, universally, it is manifest that if three straight lines be proportionals, as the first is to the third so is any rectilinear figure upon the first to a similar and similarly placed rectilinear figure upon the second.

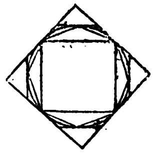
(630) Squares, like all other similar figures, are in the duplicate ratio of their sides. Hence it is usual to say, that similar figures 'are as the squares of their homologous sides;' this being only another way of expressing the duplicate ratio.

(631) The perimeters of similar rectilinear figures are as their homologous sides. For the homologous sides being those of similar triangles, are severally proportional each to each, and therefore the sum of the antecedents or the perimeter of the one polygon is to the sum of the consequents, or the perimeter of the other polygon as one antecedent is to one consequent, that is, as any two homologous sides.

(632) Since the homologous diagonals are as the homologous sides, being sides of similar triangles, it follows that the perimeters of similar rectilinear figures are as their homologous diagonals, and their areas are in the duplicate ratio of these diagonals.

Circles may be considered as similar figures and have the same properties, their diameters being esteemed diagonals. We shall then establish the two following principles.

⁂ (633) *The circumferences of circles are as their diameters or radii, and their areas are in the duplicate ratio of their diameters or radii.*



It is evident that any two regular polygons having the same number of sides are similar, and may be inscribed in circles. The radii of the circles are homologous lines in the polygons, and the perimeters of the polygons are as those radii, and their areas in the duplicate ratio of those radii. Through the vertices of the several angles of the inscribed polygons let tangents be drawn. These tangents will, if produced, form similar circumscribed polygons. By bisecting each of the arcs whose chords are the sides of the inscribed polygons, and drawing lines from the points of bisection to the angles of the polygons, inscribed polygons of double the number of sides will be obtained, and corresponding circumscribed polygons may be found in the same manner as already described. This bisection of the arcs may be continued without any limit, so that the arcs into which the circumferences are divided, as well as the sides of the polygons, will be increased in number, and diminished in magnitude *without limit*, every pair of polygons inscribed and circumscribed being always similar, and having their perimeters as the radii and their areas as the squares of the radii.

As the arcs are diminished without limit, the angles under their chords, and the tangents through the extremities of those chords, are diminished without limit; for these angles are equal to the angles in the alternate segments which stand on the arcs, and which, as the arcs are diminished without limit, will also be diminished without limit. Hence the excess of the sum of the tangents above the chord is diminished without limit, as also the area of the triangle formed by the chord and tangents. Hence by the continual bisection the excess of the perimeter of the circumscribed polygon above the perimeter of the inscribed polygon is diminished without limit, and the same may be said of their areas. Since, then, the differences of the perimeters and areas of the inscribed and circumscribed polygons may be diminished *without limit*, it follows still more evidently that the differences between either of them and the perimeters and areas of the circles (which are less than those of the circumscribed polygons and greater than those of the inscribed) may be diminished *without limit.*

Let C and C′ be the circumferences and R and R′ the radii of the circles. Then $R : R' = C : C'$; for if not, let $R : R' = C : X$, X being a line greater or less than C′.

First, let X be less than C′. Let P, P′ be the perimeters of the inscribed polygons; $R : R' = P : P'$. Hence $P : P' = C : X$; and by alternation $P : C = P' : X$. Hence, since P is less than C, P′ must be less than X. But X is (hyp.) less than C′, therefore C′ cannot exceed P′ by a magnitude less than that by which C′ exceeds X, and therefore the difference between C′ and P′ cannot be diminished without limit, contrary to what has been proved.

If X be greater than C′, let P, P′ be the perimeters of the circumscribed polygons; and in the same manner we find $P : C = P' : X$, and since P is greater than C, P′ must be greater than X; but X is greater than C′ (hyp.), and therefore the difference between P′ and C′ must be always greater than the difference between X and C′, and cannot therefore be diminished *without limit*, contrary to what has been proved.

Since then X is neither greater nor less than C′, it must be equal to C′, and therefore $R : R' = C : C'$.

If R and R′ be supposed to represent the squares of the radii, and C and C′ the areas of the circles, the same proof will establish the second part of the proposition, that the areas of circles are as the squares of their diameters.

The same reasoning which we have here applied to circles may also be applied to semicircles or any similar segments of circles, or to sectors of circles in which the central angles are equal. Hence *similar arcs are as their radii, and similar sectors or segments are as the duplicate ratio of the radii.*

⁂ (634) *The circumference of every circle bears the same ratio to its radius or diameter.*

For C : C′ = R : R′, ∵ by alternation C : R = C′ : R′. Hence if the ratio of the circumference of any one circle to its radius were known, we should be able to find a straight line equal to the circumference of a circle. This ratio, however, does not admit of being exactly expressed either by whole numbers or fractions. The radius and circumference are incommensurable lines. The results of analytical investigation, however, enable us to express the ratio with as much exactness as may be required for the most accurate practical investigations. It is found that if the diameter of a circle were divided into 100 equal parts, that 314 of these parts would be less than the circumference, and 315 greater. That if the diameter were divided into 1000 equal parts, 3141 of these parts would be less, and 3142 greater than the circumference. Again, if the number of parts of the diameter be 10000, those of the circumference will be greater than 31415, and less than 31416, and even a much greater accuracy if necessary might be obtained.

⁂ (635) *The area of a regular polygon is equal to the rectangle under the radius of the inscribed circle and its semiperimeter.*

For it may be resolved into equal isosceles triangles by lines from the centre to the angles, and the area of each triangle is equal to the rectangle under the radius and half the base, and therefore the area of the whole polygon is equal to the rectangle under the radius and half the sum of the bases or the semiperimeter.

⁂ (636) *The area of a circle is equal to the rectangle under its radius and semi-circumference.*

For the area of a polygon circumscribed round it is equal to the rectangle under the radius of the circle and the semiperimeter of the polygon. But by the continual bisection of the arcs, and the unlimited increase of the number of sides of the polygon, the difference between its perimeter and area and those of the circle may be diminished without limit, and the demonstration may be completed *ex absurdo*, as in (633).

Hence if a right line could be found equal to the circumference of a circle, a rectangle or square could be constructed equal to a circle, and the celebrated problem of 'squaring the circle' would thus be solved. The area of a circle may, however, be obtained with any proposed degree of accuracy, because the circumference may be computed with any degree of approximation (634).

The following practical rule may be derived from (634) and (636).

'To find the area of a circle multiply the square of the radius by 31415, and divide the result by 10000.' This will give the area, subject to an error of less than the 10000th part of the square of the radius.

The problem to 'square the circle,' or what is the same, to find a right line by geometrical construction equal to the circumference of the circle, has never been solved. The solution of this problem would be attended with no real advantage whatsoever, for the power of approximating numerically without limit to the circumference answers every purpose, and in fact is much more useful in practice than any geometrical construction could be. Accordingly we find at the present day no one wastes their time on these disquisitions except those whose knowledge of mathematical science is too limited to make them perceive their futility.

The degree to which the approximation has been carried by actual computation may be conceived, when we state that the circumference of a circle whose diameter is unity, has been expressed to 140 decimal places, by which the circumference of any circle can be found to within a *n*th part of the diameter, the number *n* being 1 followed by 140 ciphers. The following number expresses the circumference of a circle whose diameter is unity to 16 decimal places:

3,1415926535897932.

⁂ (637) *To construct a figure similar to a given one and bearing a given ratio to it.*

Let A be any side of the given figure, and $m : n$ be the given ratio. Let $m : n = A : B$, and let $A : B : C$. Construct the similar figure on C (621).

If the given figure be a circle, A may be its diameter. And, in general, in theorems and problems respecting similar figures when applied to circles, the radii or the chords of similar segments take the places of homologous sides.

PROPOSITION XXI. THEOREM.

(638) Rectilinear figures (A and B) which are similar to the same figure (C) are similar also to each other.

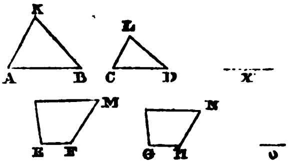
Since the rectilinear figures A and C are similar, they are equiangular, and have the sides about the equal angles proportional; and since the figures B and C are also similar, they are equiangular, and have the sides about the equal angles proportional; therefore the rectilinear figures A and B are also equiangular, and have the sides about the equal angles proportional, and are therefore similar.

PROPOSITION XXII. THEOREM.

(639) If four right lines be proportional (A B to C D as E F to G H), the similar rectilinear figures similarly described on them are also proportional.

And if four similar rectilinear figures, similarly described on four right lines, be proportional, the right lines are also proportional.

Part 1°.—Take a third proportional X to A B and C D, and a third proportional O to E F and G H; since A B is to C D as E F to G H (hyp.), C D is to X as G H to O (const.), therefore, *ex æquali*, A B is to X as E F to O; but A K B is to C L D as A B to X (XX), and E M to G N as E F to O; therefore A K B is to C L D as E M to G N.



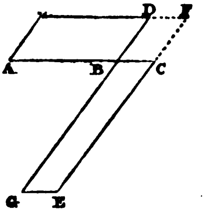
Part 2°.—Let the same construction remain: A K B is to C L D as E M to G N (hyp.), therefore A B is to X as E F to O (const.), and therefore A B is to C D as E F to G H.

This proposition is equivalent to stating that if two ratios be equal, their duplicates and subduplicates will also be equal.

PROPOSITION XXIII. THEOREM.

(640) Equiangular parallelograms (A D and C G) are to each other in a ratio compounded of the ratios of their sides.

Let two of the sides A B and B C about the equal angles be placed so that they may form one right line; since the angles A B D and D B C are equal to two right angles, and G B C is equal to A B D (hyp.), G B C and D B C are equal to two right angles, and therefore G B and B D form one right line (XIV, Book I.); complete the parallelogram B F.



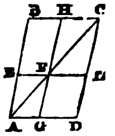
Since the parallelogram A D is to B F as A B to B C I (I), and B F to B E as B D to B G (I), A D has to B E a ratio compounded of the ratios of A B to B C, and of B D to B G.

This proposition may be immediately inferred from (I). For, since the angles are equal, the altitudes will be as the sides to which they are not perpendicular.

PROPOSITION XXIV. THEOREM.

(641) In any parallelogram (A C) the parallelograms (A F and F C) which are about the diagonal are similar to the whole and to each other.

As the parallelograms A C and A F have a common angle they are equiangular; but on account of the parallels E F and



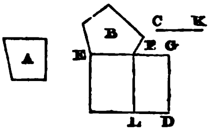
B C the triangles A E F and A B C are similar (IV), therefore A E is to E F as A B to B C; and the remaining sides are equal to A E, E F, A B, and B C, therefore the parallelograms A F and A C have the sides about the equal angles proportional, and are therefore similar.

In the same manner it can be demonstrated that the parallelograms A C and F C are similar: since, therefore, each of the parallelograms A F and F C is similar to A C, they are similar to each other.

There can be little doubt but that the places of this proposition and the succeeding one have been by some mistake transposed.

PROPOSITION XXV. PROBLEM.

(642) To construct a rectilinear figure equal to a given one (A) and similar to another (B).



On any side E F of the given figure B construct a rectangle E L equal to B (XLV, Book I.), and on the side F L construct a rectangle F D equal to A (XLV, Book I.); between the other sides E F and F G of these rectangles find a mean proportional C K (XIII); the figure described upon it, similar to the given figure B, and similarly posited, is equal to the other given figure A.

For the rectangle E L is to the rectangle F D as E F to F G (I), or in the duplicate ratio of E F to C K (const.), and therefore as the rectilinear figure B to the similar one upon C K (XX); but E L is equal to B (const.), therefore the rectilinear figure upon C K, similar to B, and similarly posited, is equal to F D, and therefore equal to the given figure A.

(643) This is one of the most important and extensively useful problems in the Elements. It may be thus announced—'To construct a figure of a given species and a given magnitude.' On the side of the figure (B) given in species a rectangle is to be constructed equal to it, and on the conterminous side of this rectangle another is to be constructed equal to the figure (A) given in magnitude. A mean proportional between the sides of these, which lie in the same right line, will be the side of the sought figure.

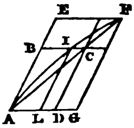
By this proposition, while a magnitude is preserved as to quantity, its shape may be changed. Thus an equilateral triangle may be reduced to a square, &c.

A figure of a given species may be found equal to the sum or difference of two magnitudes.

PROPOSITION XXVI. THEOREM.

(644) If similar and similarly posited parallelograms (A C and A F) have a common angle, they are about the same diagonal.

For, if it be possible, let A I F be the diagonal of the parallelogram A F, and draw through I the right line I L parallel to A E. Since the parallelograms A I and A F are about the same diagonal A I F, and have a common angle A, A I and A F are similar (XXIV); therefore B A is to A L as E A to A G; but B A is to A D as E A to A G (hyp.), therefore B A is to A L as B A to A D, and therefore A L is equal to A D, which is absurd. Therefore A I F is not the diagonal of A F, and in the same manner it can be demonstrated that no other line is except A C F.



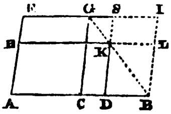
The student is recommended to omit the next three propositions, and the first solution of the thirtieth, these being at present of no use in any part of mathematical science, and inelegant and complicated both in the demonstrations and results. These propositions were frequently, however, used by the ancient geometers.

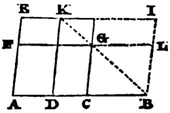
PROPOSITION XXVII. THEOREM.

(645) If any right line (A B) be bisected (in C) and cut unequally (in D), the parallelogram (F C) which is applied to the half deficient by a figure (G B) similar to itself is greater than the parallelogram (E D) applied to either of the other parts deficient by a figure (K B) similar to the former (G B).

First, let A D be the greater segment of A B, complete the parallelogram K I, and draw G B.

Since G B and K B are similar (hyp.), G B is the diagonal of both (XXVI), therefore C K is equal to K I (XLIII, Book I.), and if D L be added to both, C L is equal to D I; but C L and C E are equal (hyp.), therefore C E and D I are equal: add to both C K, and D E is equal to the gnomon C L S, and therefore less than the parallelogram C I, therefore less than F C, which is equal to C I.





Now let A D be the less segment, complete the parallelogram G I, and draw K B.

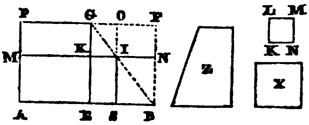
Since the parallelograms K B and G B are similar (hyp.), they are about the same diagonal (XXVI), therefore the parallelograms D G and G I are equal (XLIII, Book I.); but the right lines F G and G L are equal (hyp.), and therefore the parallelograms E G and G I are equal (XXXI, Book I.); but E G is greater than F K, therefore G I is greater than F K, and D G, which is equal to G I, is also greater than F K; add to both F D, and F C is greater than E D.

PROPOSITION XXVIII. PROBLEM.

(646) To a given right line (A B) to apply a parallelogram equal to a given rectilinear figure (Z), and deficient by a figure similar to a given parallelogram (X). But the rectilinear figure must not be greater than the parallelogram applied to half the given line, whose defect is similar to the given parallelogram (X).

Bisect A B in E, describe upon A E a parallelogram A G similar to the given one X, and complete A P F B.

A G is either equal to or greater than the given rectilinear figure Z (hyp.) If it is equal, the problem is done.



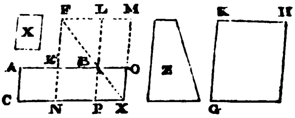
If it be greater, construct a parallelogram K L M N equal to its excess above Z and similar to X (XXV); since this parallelogram is less than A G it is less than E F, which is equal to A G (const.); but it is similar to it, and therefore its sides K L and L M are less than the homologous sides E G and G F of the parallelogram E F. Take away from these G K and G O equal to K L and L M, and complete the parallelogram K G O I; this is similar to E F since both are similar to X (const.), and it is also similarly posited, therefore K G O I and E F are about the same diagonal (XXVI); draw their diagonal G I B, produce O I to S and K I to M and N. Since the parallelogram E F is equal to the sum of K L M N and Z (const.), but K O is equal to K L M N, the gnomon E N O is equal to Z; but E I and I F are equal (XLIII, Book I.), therefore if S N be added to both E N and

S F are equal; but since A E and E B are equal, E N is equal to M E (XXXVI, Book I.), therefore M E and S F are equal, and therefore if E I be added to both M S is equal to the gnomon E N O; but E N O is equal to the given rectilinear figure Z, therefore M S is equal to the given figure Z; and its defect S N is similar to X, since it is similar to the parallelogram E F: and therefore the required problem is solved.

Proposition XXIX. Problem.

(647) To a given right line (A B) to apply a parallelogram equal to a given rectilinear figure (Z), and exceeding by a figure (B X) similar to a given parallelogram (X).

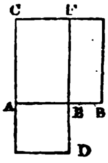
Bisect A B in E; upon E B construct a parallelogram similar to the given parallelogram X, and construct a parallelogram G H similar to the parallelogram E L, and equal to the sum of E L and Z (XXV). Since G H is greater than E L, its sides G K and K H are greater than the homologous sides of E L, which are F E and F L; on these sides produced take F N and F M equal to G K and K H, and complete the parallelogram N M; this is similar to G H, therefore similar to E L, and it is similarly placed, and therefore they are about the same diagonal; draw the diagonal F B X through A, draw A C parallel to E N until it meet P N produced. Since N M and G H are equal, and G H is equal to the sum of Z and E L, N M is also equal the sum of Z and E L; take away from both E L and the gnomon N O L is equal to Z; but since A E and E B are equal, the parallelograms A N and E P are equal, and also E P and B M are equal (195), therefore A N is equal to B M; add O N to both, and A X is equal to the gnomon N O L, and therefore equal to the given rectilinear figure Z, and its excess P O is similar to the parallelogram E L, and therefore similar to the given figure X.



Proposition XXX. Problem.

(648) To cut a given finite right line (A B) in extreme and mean ratio.

Describe B C, the square of A B (XLVI, Book I.); to A C apply a parallelogram equal to B C and exceeding by a figure A D similar to B C (XXIX); since A D is similar to B C it is a square; since B C and C D are equal, if C E be taken away from both, B F and A D are equal, and they are equiangular, therefore E F is to E D as E A to E B (XIV); but E F and E D are equal to A B and A E, therefore A B is to A E as A E to E B.



Otherwise thus :

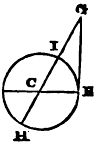
Divide A B in E so that the rectangle under A B and E B shall be equal to the square of A E (XI, Book II.), and A B is to A E as A E to E B (XVII), therefore A B is cut in extreme and mean ratio.

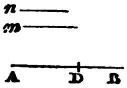
(649) If the lesser segment be taken upon the greater, the greater will be cut in extreme and mean ratio (281); and by continuing this process a series of lines will be found in continued proportion, in which the common ratio is that of the segments of a line divided in extreme and mean ratio.

The problem to divide a line in extreme and mean ratio is only a particular case of the following more general one.

⁂ (650) *To divide a line so that the rectangle under the whole line and one part shall bear a given ratio* ($m : n$) *to the square of the other part.*

Let any line E F be taken as diameter, and let a circle be described. Take a third proportional l to m and n, and upon the tangent at E take E G, a fourth proportional to m, l, and E F. Then draw G H through the centre C, and cut A B at D so that A D : D B = H I : I G, and it will be cut as required.



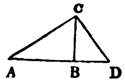


For E F : E G = $m : l$, that is, in the duplicate ratio of $m : n$, ∵ the squares of E F and E G are as $m : n$. But the square of E G is equal to the rectangle H G × G I. Therefore H G × G I : the square of H I = $m : n$. Hence H G is cut as required at I, and A B is similarly cut at D.

⁂ (651) In the solution of this problem we have assumed that *if* H I : I G = A D : D B *the rectangle* H G × G I : *the square of* H I = A B × B D : *the square of* A D. This may be easily proved. Since H I : I G = A D : D B, ∵ H G : G I = A B : B D, ∵ the rectangles H G × G I and A B × B D are similar. Also the squares of H I and A D are similar. But H I : I G = A D : D B, ∵ H I : A D = I G : D B. Therefore the similar squares on H I and A D are as the similar rectangles H G × G I, A B × B D on I G and B D. In the same manner the converse of this may be proved, *scil.* if H G × G I : the square of H I = A B × B D : the square of A D, then H I : I G = A D : D B.

⁂ (652) Hence if *two lines be cut in extreme and mean ratio they are cut similarly*, and *if a line be cut in extreme and mean ratio, any line cut similarly will be also cut in extreme and mean ratio.*

⁂ (653) *If the perpendicular* B C *in a right angled triangle divide the hypotenuse* A D *in extreme and mean ratio, the lesser side* C D *is equal to the alternate segment* A B, *and vice versâ.*



For (553) D A × B D = the square of D C, but (hyp.) it also equals the square of A B, ∵ C D = A B.

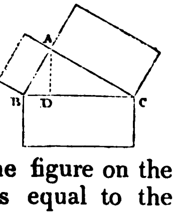
Also if C D = A B, ∵ D A × B D = the square A B, ∵ A D is cut in extreme and mean ratio at B.

⁂ (654) *The three sides of such a right angled triangle are in continued proportion, and vice versâ.* For A D × A B = the square of A C (553), but C D = A B (hyp.), ∵ A D × C D = the square of C A. Also if A D × C D = the square of A C, it also = A D × A B, ∵ C D = A B.

⁂ (655) Hence on a given hypotenuse a right angled triangle may be constructed whose sides are in continued proportion by dividing the given hypotenuse in extreme and mean ratio, describing a semicircle on it, and drawing a perpendicular to meet the semicircle, &c.

PROPOSITION XXXI. THEOREM.

(656) If any similar rectilinear figures be similarly described on the sides of a right angled triangle (B A C), the figure described on the side (B C) subtending the right angle is equal to the sum of the figures on the other side.

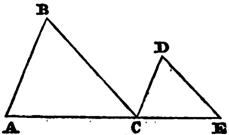


From the right angle draw a perpendicular A D to the opposite side; B C is to C A as C A to C D (550), therefore the figure upon B C is to the similar figure upon C A as B C to C D (XX), but the figure upon B C is to the similar figure upon B A as B C to B D (XX). Hence the sum of the segments B D and C D is to the hypotenuse B C as the sum of the figures on the sides is to the figure on the hypotenuse. But the sum of the segments is equal to the hypotenuse, and therefore the sum of the figures on the sides is equal to the figure on the hypotenuse.

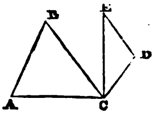
This proposition might be more immediately deduced from the twenty-second proposition of this book and the forty-seventh of the first. Any similar figures on the hypotenuse and sides are as the squares of these lines (XXII); but the sum of the squares of the sides is equal to the square of the hypotenuse, and therefore the sum of any similar figures on the sides is equal to the figure on the hypotenuse.

PROPOSITION XXXII. THEOREM.

(656) If two triangles (A B C, C D E) have two sides proportional (A B to B C as C D to D E), and be so placed at an angle that the homologous sides are parallel, the remaining sides (A C and C E) form one right line.



Because A B and C D are parallel, the alternate angles B and B C D are equal (XXIX, Book I.), and also since C B and E D are parallel, the angles D and B C D are equal (XXIX, Book I.), therefore B and D are equal; and since the sides about these angles are proportional (hyp.), the triangles A B C and C D E are equiangular (VI), therefore the angles A C B and C E D are equal; but B C D is equal to C D E, and if D C E be added, A C D and D C E are together equal to C E D, E D C, and D C E; therefore A C D and D C E are equal to two right angles (XXXII, Book I.), and therefore A C and C E form one right line (XIV, Book I.).

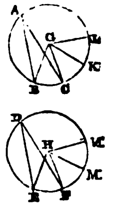


In the enunciation of this proposition, it should be stated that the proportional sides of the triangles which are not homologous should form the angle at which they are joined, for otherwise the remaining sides might not lie in the same right line. The triangles might be placed as in the annexed figure where A B : B C = C D : DE, and the sides A B and C D, as also B C and D E, are respectively parallel, but the angles A B C and C D E are supplemental, and A C and C E are obviously not in the same right line.

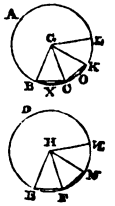
PROPOSITION XXXIII. THEOREM.

(657) In equal circles, angles, whether at the centres (B G C, E H F) or circumferences (B A C, E D F), have the same ratio which the arcs on which they stand have to one another: so also have the sectors (B G C, E B F.)

Take any number of arcs C K, K L, each equal to B C, and any number whatever F M, M N, each equal to E F: and join G K, G L, H M, H N. Because the arcs B C, C K, K L, are all equal, the angles B G C, C G K, K G L are also all



(XXVII, Book III.) equal: therefore what multiple soever the arc B L is of the arc B C, the same multiple is the angle B G L of the angle B G C: for the same reason, whatever multiple the arc E N is of the arc E F, the same multiple is the angle E H N of the angle E H F: and if the arc B L be equal to the arc E N, the angle B G L is also equal (XXVII, Book III.) to the angle E H N; and if the arc B L be greater than E N, likewise the angle B G L is greater than E H N; and if less, less: therefore as the arc B C to the arc E F, so (Def. V. Book V.) is the angle B G C to the angle E H F: but as the angle B G C is to the angle E H F, so is (XV, Book V.) the angle B A C to the angle E D F: for each is double (XX, Book III.) of each; therefore as the arc B C is to E F, so is the angle B G C to the angle E H F, and the angle B A C to the angle E D F.



Also, as the arc B C to E F, so shall the sector B G C be to the sector E H F. Join B C, C K, and in the arcs B C, C K take any points X, O, and join B X, X C, C O, O K: then, because in the triangles G B C, G C K the two sides B G, G C are equal to the two C G, G K, each to each, and that they contain equal angles, the base B C is equal (IV, Book I.) to the base C K, and the triangle G B C to the triangle G C K; and because the arc B C is equal to the arc C K, the remaining part of the circumference of the circle A B C is equal to the remaining part of the circumference of the same circle: therefore the angle B X C is equal (XXVII, Book III.) to the angle C O K; and the segment B X C is therefore similar to the segment C O K: and they are upon equal straight lines, B C, C K, and are equal (XXIV, Book III.); therefore the segment B X C is equal to the segment C O K: and the triangle B G C was proved to be equal to the triangle C G K; therefore the sector B G C is equal to the sector C G K: for the same reason, the sector K G L is equal to each of the sectors B G C, C G K: in the same manner, the sectors E H F, F H M, M H N may be proved equal to one another: therefore, what multiple soever the arc B L is of B C, the same multiple is the sector B G L of the sector B G C: and for the same reason, whatever multiple the arc E N is of E F, the same multiple is the sector E H N of the sector E H F: and if the arc B L be equal to E N, the sector B G L is equal to the sector E H N; and if the arc B L be greater than E N, the sector B G L is greater than the sector E H N; and if less, less: therefore as (Def. V. Book V.) the arc B C is to the arc E F, so is the sector B G C to the sector E H F.

By this proposition we are entitled to assume arcs referred to the same radius as measures of angles, and *vice versâ.*

(659) Every arc is to a quadrant of the same circle as the corresponding central angle is to a right angle, and it is to the whole circumference as the same angle to four right angles.

(660) Similar arcs of different circles being those which are proportional to their circumferences must subtend equal angles at the centre and circumference.

(661) Also in different circles arcs which subtend equal angles at the centre or circumference are similar.

Hence similar segments are contained by similar arcs, and *vice versâ.*

(662) *The arcs of unequal circles are in a ratio compounded of their central angles and their radii.*

Let A, A′ be the arcs, R, R′ the radii, and a, a' the angles. With a radius equal to R describe an angle equal to a', and let the subtending arc be m.

Since the arcs A and m have equal radii, ∵ $A : m = a : a'$, and since m and A′ have equal central angles they are as their radii (633), ∵ $m : A' = R : R'$. But $A : A'$ is a ratio compounded of the ratios $A : m$ and $m : A'$, or of the equivalent ratios $a : a'$ and $R : R'$.

(663) *Central angles are in a ratio compounded of the direct ratio of their arcs, and the inverse ratio of their radii.*

For by (662) we have $A : A' = \left\{\begin{array}{l} a : a'.\\ R : R'.\end{array}\right.$

Let each of these equal ratios be compounded with the ratio $R' : R$, and we have

$$\left.\begin{array}{l} A : A\\ R' : R\end{array}\right\} = \left\{\begin{array}{l} a : a'.\\ R : R'.\\ R' : R.\end{array}\right.$$

But $\left.\begin{array}{l} R : R'\\ R' : R\end{array}\right\}$ is a ratio of equality (597), ∵

$$\left.\begin{array}{l} A : A'\\ R' : R\end{array}\right\} = a : a'.$$

THE
ELEMENTS OF SOLID GEOMETRY.

INTRODUCTION.

(1.) THE first six books of Euclid's Elements, to which we have directed the attention of the student in the preceding part of this volume, are confined to the investigation of the properties of rectilinear figures and circles which are all described upon the *same plane*. It is evident that it may be, and very frequently is, necessary to consider the mutual relations and properties of right lines and circles which are in different planes, and also the various circumstances which regulate the relative position of planes themselves. Besides this, there are numerous other surfaces on which, as well as on planes, right lines or circles, or both may be drawn. The properties of such surfaces, and the various lines which may be described upon them, form an important part of geometrical science. But even this gives a very inadequate notion of the infinite extent of the field which geometry presents to our contemplation. The right line and circle are the most simple of all lines, and those which perhaps most frequently become the subjects of examination; but they are far from including the whole even of that class of lines which are described upon a plane, not to mention innumerable curves which are drawn upon other surfaces, but which are not all in the same plane. The variety of surfaces is as infinite as that of lines. They are divided into *plane* and *curved* surfaces. All *plane surfaces* are perfectly alike in their properties, but *curved surfaces* admit of endless variety.

Of this extensive field for geometrical investigation, long-established usage has assigned a certain part to 'the elements of geometry.' The 'elements of plane geometry' are confined to the properties of right lines and circles described *upon the same plane*, excluding ellipses, hyperbolas, and innumerable other lines, which in common with the right line and circle admit of being drawn upon a plane, and are thence called 'plane curves.' These are generally assigned to the province of the sublimer geometry, and their properties are investigated most easily and effectually by the powerful instruments of analysis. The student who desires to prosecute such investigations is referred to our treatise on Analytic Geometry.

The 'elements of solid geometry' are confined to the investigation of the circumstances which determine the mutual position of right lines and circles which are not in the same plane, the properties of solid figures, which are bounded by planes, and those of the surfaces denominated *spheres, cylinders*, and *cones*, and the solids bounded by these alone, or by these conjointly with planes. The unlimited variety of curved surfaces which do not come under these denominations are resigned to the province of the higher geometry, and like the 'plane curves' already mentioned are brought under the dominion of analysis. The student who desires to penetrate to the depths of this department of the science, will find ample information and assistance in the beautiful work of MONGE, entitled *Application d'Algèbre à la Géométrie.*

Conformably to what we have now stated, we shall devote the present treatise to the investigation of the conditions which determine the mutual position of right lines which are not in the same plane, of different planes with respect to each other and to right lines, the properties of figures or spaces bounded by planes, and the principal properties of spheres, cylinders, and cones.

BOOK I.

Of the Relative Position of Right Lines and Planes.

(2) Def.—A plane is a surface such that a right line cannot be drawn through two points in it without having all its points in the surface.

There are other surfaces besides a plane in which it is *possible* to assume two points such that if a right line be drawn through them, and be indefinitely produced, it will lie entirely in the surface, but in a plane surface it is *impossible* to assume two points with which this will not happen. This is not true of any other surface.
(3) Cor.—Hence it follows, that one part of a right line cannot be in a plane while another part of it is above or below it.

Proposition I.

(4) If two planes cut each other, their common intersection will be a right line.

For if any two points of their common intersection be assumed, and a right line be drawn through them, this right line must lie entirely in each of the planes (2), and must therefore be their common intersection.

(5) Def.—A plane is said to be *drawn through* a right line when it is drawn through two points of that line. The whole line will in this case be in the plane.

(6) It is evident that innumerable planes may be drawn through the same right line, or what is the same, any number of planes may intersect each other in the same right line. This will easily be perceived if any plane drawn through the right line be conceived to revolve round that right line. The different positions which it will assume in different parts of its revolution will be those of different planes drawn through the right line.

PROPOSITION II.

(7) Two planes can have only one line of intersection.

For suppose that they had a second. Through any two points on those lines of intersection let a right line be drawn. By (2) every part of this line is in each of the planes. Therefore it is a third line of intersection; and the same being true of right lines drawn through every two points on the lines of intersection, it follows that every right line which is drawn in one plane is also in the other, and therefore the two planes are identical.

Hence two distinct planes cannot have more than one line of intersection.

This proposition is analogous to that in virtue of which two right lines can intersect only in one point.

PROPOSITION III.

(8) If a point be given, and also a right line not passing through the given point, a plane may be drawn through them, and but one such plane can be drawn.

Let a plane be drawn through the given right line, and being indefinitely produced, let it be conceived to revolve round that right line. In its revolution it must sweep through all the surrounding space, and must therefore pass through the given point.

There is but one plane passing through the given right line which also passes through the given point; for if we were to suppose a second plane it would evidently have two intersections with the first, *viz.* the given right line and another intersection passing through the given point.

The student should recollect that planes are considered as indefinitely produced.

(9) COR.—Hence a right line and a point, provided the point be not on the right line, are sufficient to determine a plane.

PROPOSITION IV.

(10) A plane, and but one plane, can be drawn through three points which are not on the same right line.

Let a right line be drawn through any two of the points, and a plane, and but one plane, can be drawn through this line and the third point (8).

(11) Cor.—Hence three points, not placed in the same right line, are sufficient to determine a plane.

Proposition V.

(12) A plane, and but one plane, can be drawn through two intersecting right lines.

For let a point be assumed on each of them different from their point of intersection. A plane, and but one plane, can be drawn through the two assumed points and the point of intersection (10), and the two intersecting lines will be in this plane (2).

(13) Cor.—Hence two right lines which intersect are sufficient to determine a plane.

(14) Def.—The plane which is drawn through two intersecting lines is usually called 'the plane of those lines,' or 'the plane of the angle' which those lines contain.

Proposition VI.

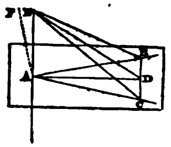
(15) A plane, and but one plane, can be drawn through two parallel lines.

A plane may be drawn through them because by their definition they are in the same plane; and but one plane can be drawn through them, because but one plane can be drawn through either of them, and any point assumed upon the other (8).

Proposition VII.

(16) If two right lines intersect, a third right line may be drawn through their point of intersection perpendicular to each of them.

Let A B and A C be the right lines intersecting at A. Take equal parts A B, A C from A and draw B C. Bisect B C at D, and draw a line D E perpendicular to B C, and making any angle with A D, and let the acute angle be A D E. Through A and in the plane of the lines A D E (XIII) draw A E perpendicular to A D. The line A E will then be also perpendicular to A B and A C. For draw E C and E B.



Since B A C is an isosceles triangle and A D bisects B C, A D is perpendicular to B C (El. (75)).* Hence A D C and A D B are right angled triangles. Also E D C and E D B are right angles. Hence the square of E C is equal to the sum of the squares of E D and D C. But also because of the right angle E A D, the square of E D is equal to the sum of the squares of E A and A D. Hence the square of E C is equal to the sum of the squares of E D, A D, and D C. But because of the right angle A D C the square of A C is equal to the sum of the squares of A D and D C. Hence the square of E C is equal to the sum of the squares of E A and A C, and therefore the angle E A C is right.

In the same manner it may be proved that the angle E A B is right.

Proposition VIII.

(17) A right line which is perpendicular to two right lines which intersect, is also perpendicular to every line in their plane drawn through their point of intersection.

For let A E be perpendicular to the lines A B and A C, and let A D be any other line through A in the plane of the angle B A C. Let any line B C be drawn intersecting the sides A B, A C, and bisected by the line A D, and draw E B, E C.

In the triangle B E C, since the line E D bisects the base B C, the sum of the squares of E B and E C is equal to twice the sum of the squares of E D and D C (El. (299)). Also because of the right angled triangles B A E and C A E, the square of B E is equal to the sum of the squares of B A and A E, and the square of C E is equal to the sum of the squares of C A and A E. Hence the sum of the squares of B E and C E is equal to the sum of the squares of B A and C A together with twice the square of E A. Hence twice the sum of the squares of D E and D C is equal to the sum of the squares of B A and A C together with twice the square of A E.

But since the line A D bisects B C, the sum of the squares of B A and C A is equal to twice the sum of the squares of D A and D C. Hence twice the sum of the squares of D E and D C is equal to twice the sum of the squares of E A, A D, and D C. Take from both twice the square of D C, and we find that twice the sum of the squares of E A and A D is equal to twice the square of E D, and therefore the sum of the squares of E A and D A is equal to the square of E D, and therefore the angle E A D is right; and the same may be proved of any other right line drawn through A in the plane of the angle B A C.

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* Where El. precedes a reference, the article referred to is in the first six books of Euclid's Elements; otherwise the reference is to the Solid Geometry.

Hence the line A E is perpendicular to every right line drawn through A in the plane of the angle B A C.

(18) Def.—A right line, such as A E, which is drawn from a point in a plane so as to be perpendicular to all lines drawn in the plane through that point, is said to be perpendicular to the plane itself.

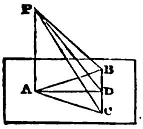
(19) Cor.—Through the same point A in a given plane B A C only one right line A E can be drawn perpendicular to the plane. For if another could be drawn let it be A F, and let A C be the intersection of the plane of the angle F A E with the given plane B A C. Since F A and E A are both perpendicular to the plane B A C, the angles E A C and F A C are both right, and therefore equal, a part to the whole, which is absurd.

(20) Def.—We shall call that point at which a perpendicular to a plane meets the plane, *the foot of the perpendicular.*

Proposition IX.

(21) From a given point out of a given plane a right line may be drawn perpendicular to the plane, and only one such line can be drawn.

Let A B C be the given plane, and P the given point. Draw any line P B from P to the plane; if this be perpendicular to the plane, the proposition is true. If not, let any line B C be drawn from the point B, and in the given plane, and let the angle P B C be acute. From P inflect on B C a right line P C, equal in length to P B, so that the triangle B P C shall be isosceles. Bisect B C at D, and in the given plane draw D A perpendicular to B C, and from P draw P A perpendicular to A D. This line P A will be perpendicular to the given plane.



For draw A B, A C, and P D.

Since B P C is an isosceles triangle and P D bisects the base, it is perpendicular to the base, and the angles P D B, P D C are right. The angles A D C and A D B are right by construction. The square of P B is equal to the sum of the squares of P D and D B. But since P A D is right by construction, the square of P D is equal to the sum of the squares of P A and A D. Hence the square of P B is equal to the sum of the squares of P A, A D, and B D. But since A D B is a right angle, the sum of the squares of A D and B D is equal to the square of A B. Hence the square of P B is equal to the sum of the squares of P A and A B, and therefore the angle P A B is right, and since P A D is also right, the line P A is perpendicular to the plane A B C (17, 18).

It is evident that only one perpendicular can be drawn from the same point P, because if two were supposed to be drawn they would both be perpendicular to the line joining the points where they would meet the plane, and thus two right angles would be in the same triangle.

PROPOSITION X.

(22) Of the several lines which may be drawn from a given point P to a given plane A B C,

1°. The perpendicular P A is the shortest.

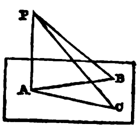
2°. Those which are equally inclined to the perpendicular are equal, and *vice versâ.*

3°. The greater the angle which a line makes with the perpendicular, the greater the line is, and *vice versâ.*

4°. Lines which meet the plane at equal distances from the foot of the perpendicular are equal, and *vice versâ.*

5°. The more distant the point where a line meets the plane is from the foot of the perpendicular, the greater is the line, and *vice versâ.*

1°. The perpendicular P A is the shortest line because it is the side of a right angled triangle P A B, of which any other line P B is the hypotenuse.



2°. If the angles C P A and B P A be equal, since the angles P A B and P A C are right, and P A common to the triangles C P A and B P A, the sides B P and P C are equal.

If the sides P B and P C be equal, since the angles P A B and P A C are right and P A common, the angles A P B and A P C must be equal.

3°. If the angle A P B be greater than the angle A P C, since the side P A is common and the angles at A right, the side A B must be greater than A C, and therefore P B greater than P C.

If P B be greater than P C, since P A is common and the angles at A right, the side A B is greater than A C, and therefore the angle A P B greater than the angle A P C.

4°. If A B be equal to A C, since A P is common, and the angles at A right, the lines P B and P C must be equal.

If P B be equal to P C, since P A is common, and the angles at A right, the sides A B and A C are equal.

5°. If A B be greater than A C, since P A is common, and the angles at A right, the line P B must be greater than P C.

If P B be greater than P C, since P A is common, and the angles at A right, the line A B must be greater than A C.

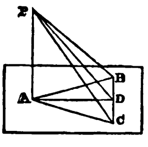
(23) Cor.—Hence all equal oblique lines from a point P to a given plane terminate in the circumference of a circle described upon the plane of which the foot of the perpendicular is the centre; and also all lines which drawn from the same point to a plane meet the plane in the circumference of a circle, of which the foot of the perpendicular is the centre, are equal.

Proposition XI.

(24) If P A be perpendicular to the plane A B C, and B C be a right line in that plane, and from A A D be drawn perpendicular to B C, then P D will be perpendicular to B C.

On each side of D take equal parts D B and D C, and draw P B, P C, A B, and A C.

Since D B is equal to D C, and the angles at D are right, A B is equal to A C. Also since A B and A C are equal, and the angles P A B and P A C are right, P B and P C are equal. In the isosceles triangle B P C, P D bisects the base and is therefore perpendicular to it.

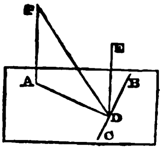


(25) Cor.—It is evident that B C is perpendicular to the plane P D A since it is perpendicular to D P and D A.

(26) The lines P A and B C are an instance of two lines which, without being parallel, can never meet if produced indefinitely, since the same plane cannot be drawn through them.

Proposition XII.

(27) If a right line be perpendicular to a plane, every right line which is parallel to it is perpendicular to the same plane.



Let P A be perpendicular to the plane A B C, and let D E be parallel to P A. Draw A D, and draw B C perpendicular to A D, and in the given plane and from any point P in the perpendicular A P draw P D.

The line B C is perpendicular to the plane P D A (25), and since D E is parallel to P A the same plane may be drawn through them, and this plane is evidently that of the angle E D A. Since B C is perpendicular to the plane P D A or E D A, E D C is a right angle. But because of the parallels, E D A is a right angle. Hence E D is perpendicular to D C and D A, and is therefore perpendicular to the plane A B C.

PROPOSITION XIII.

(28) Perpendiculars to the same plane are parallel.

For if two perpendiculars be not parallel, through the foot of one draw a parallel to the other. This will be perpendicular to the plane (27), and therefore two perpendiculars to the same plane would pass through the same point, which cannot be (19).

PROPOSITION XIV.

(29) Right lines which are parallel to the same right line are parallel to each other.

For the plane which is perpendicular to the last will be also perpendicular to the others (27), and therefore the other lines must be parallel to each other (28).

This proposition applied to parallels in the same plane is the thirtieth proposition of the first book of the Elements.

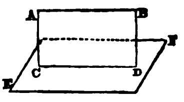
(30) DEF.—When a plane and a right line are so placed that each being indefinitely produced they will never meet, they are said to be parallel.

PROPOSITION XV.

(31) If two right lines be parallel, every plane drawn through one of them is parallel to the other.

(The plane of the parallels themselves is here excepted.)

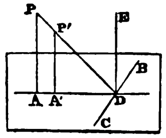
Let A B and C D be the parallels, and let E F be a plane drawn through C D. The line A B is parallel to the plane E F.



For let A D be the plane of the parallels. The line A B, however produced, cannot pass out of the plane A D, and therefore cannot meet the plane E F without meeting C D; but it is parallel to C D, and therefore can never meet the plane E F, and is therefore parallel to it.

(32) Def.—The inclination of a right line to a plane is the complement of the inclination of that right line to a perpendicular to the plane drawn through the point where the line meets the plane.

(33) If from any point P in a right line P D which intersects a plane A B C at D, a perpendicular P A be drawn, and also the line D A, the inclination of P D to the plane A B C is equal to the angle P D A.



For through D draw D E perpendicular to the plane. The lines P A D E are in the same plane, and therefore the angle P D A is the complement of P D E, and is therefore equal to the inclination of the line P D to the plane.

From whatever point of the line P D the perpendicular be drawn, it will meet the plane in the same right line A D. Suppose it drawn from P′. The lines P A and P′ A′ being parallel (28) are in the same plane, and that plane is evidently the plane of the angle P D A, or P D E.

(34) Def.—If perpendiculars P A, P′ A′ be drawn from the extremities of a right line P P′ to a plane, the right line A A′ joining the feet of the perpendiculars is called the *projection* of the right line P P′ upon the plane.

(35) Cor. 1.—Hence it appears that the inclination of a right line to a plane is equal to the inclination of that right line to its *projection* on the plane.

(36) Cor. 2.—If a right line be parallel to a plane it is parallel to its *projection*, and therefore equal to it, but otherwise the line is always greater than its projection.

The relation between a line and its projection is expressed trigonometrically thus: $a = p$ cos. A, where a is the line, p its projection, and A its inclination.

(37) Def.—Two planes which being indefinitely produced never meet, are said to be parallel.

Proposition XVI.

(38) Planes which are perpendicular to the same right line are parallel.

For if not, let them meet, and from the feet of the perpendicular draw two right lines to any point of their intersection. These lines will both be at right angles to the perpendicular (18), so that the triangle thus formed will have two right angles, which is absurd.

Proposition XVII.

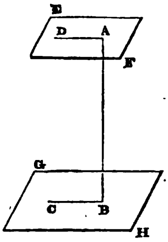
(39) If two parallel planes be intersected by a third plane their common intersections are parallel.

For they are in the same plane, as is evident, and they can never meet, for if they did the parallel planes in which they respectively are would meet.

Proposition XVIII.

(40) If two planes be parallel, any right line which is perpendicular to one will also be perpendicular to the other.

Let A B be perpendicular to the plane E F, and intersect the plane G H which is parallel to E F at B. Through A B let any plane be drawn intersecting the parallel planes in A D and B C. The lines A D and B C are by hyp. in the same plane, and since they are also in parallel planes they can never meet, and are therefore parallel. Hence the angles B A D and A B C are supplemental; but B A D is right by hyp. and therefore A B C is also right, and the same being true for every plane drawn through A B, the line A B is perpendicular to every line through B in the plane G H, and is therefore perpendicular to the plane itself.



Proposition XIX.

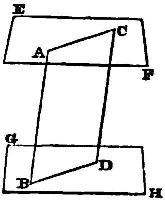
(41) Planes which are parallel to the same plane are parallel to each other.

For let a right line be drawn perpendicular to the latter. It is also perpendicular to the former (40), and therefore the former are parallel (38).

PROPOSITION XX.

(42) The parts of parallel lines intercepted between parallel planes are equal.

Let E F and G H be the parallel planes, and A B and C D the parallel lines. Draw A C and B D.



Since A C and B D are in parallel planes they cannot meet, and since also they are in the same plane (that of the parallels) they are parallel. Hence A D is a parallelogram, and therefore A B and C D are equal.

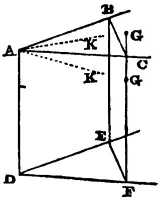
(43) Cor.—Hence all perpendiculars drawn between parallel planes are equal, and therefore parallel planes are equidistant.

PROPOSITION XXI.

(44) If two angles have their sides respectively parallel and lying in the same direction, they will be equal, and their planes will be parallel.

If they be in the same plane, the proposition has been proved in the Elements (117).

If they be not in the same plane, let the angles be B A C and E D F.



Take A B equal to D E and A C to D F, and draw the lines B E F C.

Since A B and D E are equal and parallel, B E and A D are equal and parallel. In the same manner C F and A D are equal and parallel, and therefore B E and C F are equal and parallel, and hence B C and E F are equal and parallel. The triangles B A C and E D F are therefore mutually equilateral, and therefore mutually equiangular, and hence the angle B A C is equal to the angle E D F.

The planes B A C and E D F are parallel, for if not draw through A a plane parallel to E D F, and let it intersect C F at any point G different from C. It follows from (42) that G F is equal to A D, but C F has been proved equal to A D, and therefore G F is equal to C F, which is absurd.

PROPOSITION XXII.

(45) **If the planes of two equal angles (B A C and E D F) and one pair of their legs (A B and D E) be parallel, and the other pair of legs (A C and D F) be at the same side of the parallel legs, they will also be parallel.**

For if A C be not parallel to D F, draw A K parallel to D F. Hence the angle K A B is equal to F D E (44), and therefore to C A B, which is absurd.

(46) DEF.—The inclination of two right lines which do not meet, or through which the same plane cannot be drawn, is estimated by the angle contained by any two lines drawn through the same point parallel to them. It appears from the preceding proposition that this angle will be the same whatever be the point through which the lines which form it are drawn.

PROPOSITION XXIII.

(47) **If three right lines be parallel, equal, and not in the same plane, the triangles formed by right lines joining their extremities are mutually equilateral, and their planes are parallel.**

Let the parallel lines be A D, B E, and C F. Since B E and A D are equal and parallel, A B and D E are equal and parallel, and in the same manner it may be proved that A C and D F, B C and E F are equal and parallel. Hence the triangles A B C and D E F are mutually equilateral, and it may be proved as in (44) that their planes are parallel.

(48) DEF.—The inclination of two planes, or the angle under them, is estimated by the inclination of two right lines which are respectively perpendicular to them.

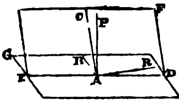
Since all perpendiculars to the same plane are parallel, it follows that the angle by which the inclination of two planes is estimated must be the same, wherever the two perpendiculars to the planes may be drawn. This will plainly appear from (44) and (45). By this definition the mutual inclination of planes is determined by that of right lines, which considerably simplifies all investigations respecting them.

Hence the student will easily perceive that the common properties of intersecting right lines may be extended to intersecting planes. If a plane intersect another the adjacent angles are together equal to two right angles, and the angles vertically opposite are equal. Also if one plane be perpendicular to another, the latter is perpendicular to the former. Also, if a plane intersect two parallel planes, consequences may be deduced similar to those in the twenty-seventh, twenty-eighth, and twenty-ninth propositions of the first book of the Elements.

PROPOSITION XXIV.

(49) If two planes intersect, their mutual inclination is equal to the angle contained by right lines drawn from the same point on their common intersection in the planes and respectively perpendicular to their intersection.

Let E F and G H be the planes and E D their intersection. From any point A draw A B perpendicular to E D, and in the plane G H and A C perpendicular to E D and in the plane E F. The angle B A C is the inclination of the planes.



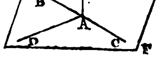
For from A draw A P perpendicular to G H and A R perpendicular to E F. Since R A D and P A D are right angles, the plane of the angle P A R is perpendicular to the line A D, and since C A D and B A D are right angles, the plane of the angle B A C is at right angles to A D. Hence the angles B A C and P A R are in the same plane. Since A R is perpendicular to E F, the angle R A C is right, and since P A is perpendicular to G H, the angle P A B is right. Hence the angle R A C is equal to the angle P A B. From both take the common part P A C and the remainders are equal, that is, P A R, which is the inclination of the planes, is equal to the angle B A C.

PROPOSITION XXV.

(50) If a right line P A be perpendicular to a plane E F, every plane drawn through the perpendicular P A is also perpendicular to E F.

Let P A C be a plane drawn through the perpendicular P A, and from A in the plane E F draw A D perpendicular to A C.

Since P A is perpendicular to the plane E F, the angles P A C and P A D are right, and therefore the line D A, being perpendicular to A P and



A C, must be perpendicular to the plane P A C (17, 18). Since then A D is perpendicular to the plane P A C, and A P is perpendicular to the plane E F, the inclination of those planes is that of the lines A D and A P. But P A D is a right angle, and therefore the planes are perpendicular, and the same may be proved of any plane drawn through P A.

PROPOSITION XXVI.

(51) If a plane A P C be perpendicular to another E F, and if the line P A be drawn in the plane A P C perpendicular to the line of intersection B C, then P A will be perpendicular to the plane E F.

For draw A D in the plane E F and perpendicular to B C. The angle P A D is the inclination of the two planes (48), and is therefore a right angle. But P A C is a right angle by hyp. Hence the line P A being perpendicular to two lines in the plane E F is perpendicular to the plane E F (17, 18).

(52) COR.—It is evident that if from the intersection of two perpendicular planes a right line be drawn perpendicular to either, it will be entirely in the other.

PROPOSITION XXVII.

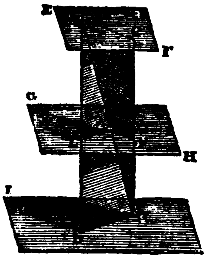
(53) If two planes be perpendicular to a third plane, their common intersection will be perpendicular to the third plane.

For if, from the point where their common intersection meets the third plane, a perpendicular to the third plane be drawn, that perpendicular must be in each of the two planes (51), and must therefore be their intersection.

PROPOSITION XXVIII.

(54) Right lines intersecting parallel planes are divided proportionally.

Let the parallel planes be E F, G H, and I K, and let the right lines which intersect them be A B and C D. Draw A D and



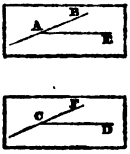
join the several points where the right lines meet the planes by the lines A C, L M N, and B D.

In the triangle B A D since L M and B D are at the same time in parallel planes, and in the same plane, they must be parallel, for they cannot meet. In the same way A C and M N are parallel. Hence by the two triangles

we have A L : L B :: A M : M D,
C N : N D :: A M : M D,
therefore A L : L B :: C N : N D.

Proposition XXIX.

(55) If two right lines be not in the same plane, planes may be drawn through them which are parallel, and only two such planes can be drawn.



Let the lines be A B and C D. Through any point A of the line A B draw A E parallel to C D, and through any point C of the line C D draw C F parallel to A B. The planes of the angles B A E and F C D are parallel (44). It is evident that no other parallel planes can be drawn through A B and C D.

Proposition XXX.

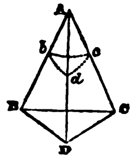
(56) If two right lines be not in the same plane, a third right line may be drawn intersecting them at right angles, and only one such line can be drawn.

Let parallel planes be drawn through the given right lines, and also let planes be drawn through each of them at right angles to those parallel planes. The intersection of these two planes will evidently intersect the given right lines at right angles, and will be the only line which can be so drawn.

Proposition XXXI.

(57) The right line which intersects perpendicularly two right lines not in the same plane, is the shortest line which can be drawn between those two right lines.

For if it is perpendicular to the parallel planes which may be drawn through them, any other line drawn between them would be oblique to these planes and therefore longer.



(58) Def.—A solid angle is formed by three or more planes which meet at the same point. Thus the planes B A C, B A D, and D A C form a solid angle.

(59) Def.—The point A where the planes meet is called the *vertex* of the solid angle.

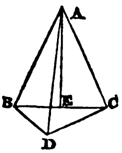
It is evident that the sides or faces of a solid angle are plane angles, and that less than three plane angles cannot form a solid angle.

(60) Def.—The *edges* of a solid angle are the lines (A B, A C, A D) in which the plane angles intersect.

If with the point A as centre, and any distance A *b* as radius, a circular arc *b d* be described in the plane of the angle B A D, and another *b c* in the plane of the angle B A C, and a third *d c* in the plane of the angle D A C, a triangle *b d c* will be formed by the three arcs, called a spherical triangle. The sides of this triangle are the measures of the plane angles which form the solid angle A, and its angles are the inclinations of the planes of these angles. The properties of solid angles are thus identified with those of spherical triangles, and they form the subject of spherical geometry and trigonometry. On this subject the student is referred to the second part of my treatise on trigonometry. If the solid angle be formed by more than three plane angles, it corresponds to a spherical figure with more than three sides. In spherical geometry the only property of a solid angle which has been borrowed from solid geometry is that which is established in the following proposition.

Proposition XXXII.

(61) If a solid angle be formed by three plane angles, any two of these taken together must be greater than the third.



It is only necessary to prove that the greatest of the three plane angles is less than the sum of the other two. Let B A C be the greatest, and draw A E so that the angle C A E shall be equal to the angle C A D. On the line A D take A D equal to A E, and draw B D and D C.

In the triangles C A D and C A E the sides A D and A E are equal, A C is common, and the angles C A D and C A E are equal, therefore the bases C E and C D are equal. Hence B E is the difference of the sides B C and C D of the triangle B C D, and is therefore less than the base B D. In the triangles D A B and E A B the sides D A and E A are equal, and B A is common, and the base B D has been proved greater than B E, and therefore the angle B A D is greater than the angle B A E. To these let the equal angles C A D and C A E be added, and it follows that the sum of the angles B A D and C A D is greater than the angle B A C.

(62) All the other properties of solid angles may be deduced from the results of spherical geometry and trigonometry. Thus we find that in a solid angle formed by several plane angles, any one of the plane angles is less than the sum of all the others. That 'the sum of the plane angles which form any solid angle must be less than four right angles.' TRIG. (130).

That 'if a solid angle be formed by three plane angles the sum of the inclinations of the planes cannot be less than two right angles nor greater than six,' but may have any intermediate magnitude. TRIG. (140).

These and other properties too numerous to insert here will be found in the work already cited. It may, however, be worth mentioning, that of the six quantities related to a solid angle contained by three planes, *viz.* the three plane angles and the three inclinations of the planes, any three being given the other three can always be determined. TRIG. Part II. Sect. VII.

BOOK II.

Of Solid Figures which are bounded by Planes.

(63) SINCE three planes are necessary to form a solid angle, it is evident that they cannot completely enclose a solid space. There will be in one direction a void which cannot be closed without one additional plane at least. Hence it appears, that less than four planes cannot enclose solid space, and therefore a solid figure cannot have less than four *faces*, the plane figures which enclose such a solid being called *faces*. The solid may also be conceived to be bounded by the right lines formed by the intersections of the planes of its faces. These are called its *edges;* and as every distinct pair of faces has a distinct intersection, a solid figure will have as many *edges* as it has distinct pairs of *faces*. By the principles of algebra, it follows, that if n be the number of bounding planes or *faces*, $\frac{n \,.\, n-1}{1 \,.\, 2}$ is the number of edges. Thus if the number of faces be 4, the number of edges is $\frac{4 \times 3}{2}$ or 6; if the number of faces be 5, the number of edges is $\frac{5 \times 4}{2}$ or 10; if the number of faces be 6, the number of edges is $\frac{6 \times 5}{2}$ or 15. In this manner we may construct a table showing the number of edges corresponding to any given number of faces :

Faces	4	5	6	7	8	9	10
Edges	6	10	15	21	28	36	45

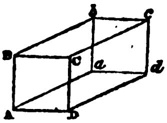
which may easily be continued to any number.

(64) Solid figures receive denominations expressive of the number of their faces; thus a figure with *four* faces is called a *tetraedron,* one with *six* faces an *hexaedron,* and so on. Generally, solids with more than six faces are called *polyedrons.*

(65) Solids also receive denominations according to the figures and position of their faces, as in the following instances.

(66) DEF.—A *prism* is a solid figure two of whose faces are equal and similar rectilinear figures so placed that their equal sides are respectively parallel, the other faces being parallelograms formed by right lines joining the vertices of the corresponding angles of these rectilinear figures. These figures are called the *bases* of the prism, and the edges formed by the right lines which are drawn connecting the vertices are called the *sides* of the prism.

Let A B C D and *a b c d* be equal and similar rectilinear figures described upon parallel planes. And let A B and *a b*, two homologous sides, be parallel, and so placed that the vertices A and *a* of corresponding angles will be opposite. It will then follow that all the other homologous sides of the figures will be parallel each to each. For since A B and *a b* are parallel, and also the planes of the angles B A D and *b a d*, and these angles themselves are equal, it follows that the sides A D and *a d* are parallel (45); and the same may be proved successively of each pair of homologous sides.



Since A B and *a b* are equal and parallel, the figure A B *b a* is a parallelogram, and in like manner it may be shown that the other faces formed by the lines joining the corresponding vertices of the bases are parallelograms.

It is evident that all the *sides* of a prism are equal.

(67) Def.—The *altitude* of a prism is the perpendicular distance between its bases.

(68) Def.—A prism is said to be *right* or *oblique*, according as its sides are perpendicular or oblique to its bases.

(69) Def.—Prisms are denominated from the nature of their bases, *triangular*, *quadrangular*, *pentagonal*, &c.

(70) Def.—A prism whose bases are parallelograms is called a *parallelopiped*.

A parallelopiped is therefore an hexaedron all whose faces are parallelograms, and each pair of faces which do not actually intersect are parallel. Any two parallel faces may be taken as the bases of the prism.

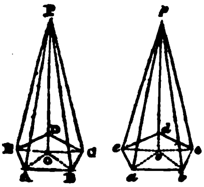
If the bases of a parallelopiped be rectangles and its sides be perpendicular to them, all the faces will evidently be rectangles. In this case it is called a *rectangular parallelopiped*.

(71) Def.—If the bases of a rectangular parallelopiped be squares, and the altitude be equal to the side of the base, all its faces will be squares. Such a parallelopiped is called a *cube*.

Proposition I.

(72) If the bases of two prisms be equal and similar, and two homologous sides of the bases be equally inclined to the sides of the prisms with which they form a solid angle, the several sides of each prism will be inclined to the sides of the base which they meet at angles which are respectively equal.

Let the base *a b c d e* be placed upon the base A B C D E, so that the several homologous sides shall coincide. Let the side *b g* be inclined to *b a* and *b c*, at the same angles as B G is inclined to B A and B C. Hence the side *b g* coincides with B G. Since the point *a* coincides with A and the lines B A F and *b a f* are in the same plane, and the angles B A F and *b a f* are equal, the line *a f* must coincide with A F; and in the same manner it may be proved that the several sides of the prism whose base is *a b c d e* will coincide with the corresponding sides of the other prism, and therefore the angles under these sides and those of the base are respectively equal to each.



Proposition II.

(73) **If two prisms have equal bases and one pair of corresponding sides equal and similarly inclined to the sides of the bases with which they form solid angles, the prisms will be equal in every respect.**

For by the demonstration of (72) it appears, that the base of one may be so applied to the base of the other that the several sides of the one will respectively coincide with the sides of the other; and since these sides are equal the opposite bases must coincide, and therefore the several vertices of the one prism will coincide with those of the other, and the two solids will fill exactly the same spaces and be bounded by the same lines and planes.

(74) Cor.—Hence it obviously follows, that right prisms, which have equal and similar bases and equal altitudes, are equal in all respects.

Proposition III.

(75) If two parallelopipeds have three conterminous edges in the one equal to three conterminous edges in the other and including angles which are equal each to each, the parallelopipeds are equal in all respects.

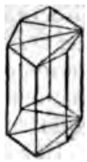
For if two conterminous edges in one be equal to two in the other, the faces of which these edges are sides will be equal, and thus the proposition becomes a particular case of (73).

(76) Cor.—If the vertex of a solid angle of a parallelopiped

be given in position, and the three edges terminated at that vertex be given in magnitude and position, the parallelopiped is given.

PROPOSITION IV.

(77) **Every prism may be divided into as many triangular prisms as there are triangles into which its base may be resolved by diagonals drawn from the vertex of any of its angles.**

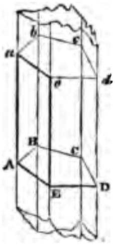


Since each pair of sides are equal and parallel, it follows that the diagonals of the bases which connect the extremities of the sides are equal and parallel, and the figure formed by the sides and diagonal is therefore a parallelogram. There are as many of these parallelograms, which we shall call diagonal planes, as there are different diagonals of the bases of the prism; and it is evident that the prism may be resolved into triangular prisms by diagonal planes, all of which pass through any one side and severally through the other sides, except those which are adjacent to that side which is their common intersection. This will be evident on inspecting the figure.

(78) Cor.—It is evident that each diagonal plane is parallel to the sides of the prism, and also that the intersection of any two such planes is parallel to the sides.

PROPOSITION V.

(79) **The sections of a prism by parallel planes are similar and equal rectilinear figures.**



Let A B C D E and *a b c d e* be two parallel sections. Since A B and *a b* are the intersections of parallel planes with the same plane they are parallel, therefore A *a b* B is a parallelogram, and therefore A B and *a b* are equal. In the same manner it may be proved that B C is equal to *b c*, C D to *c d*, and so on.

Since the sides of the angle A B C are parallel to those of *a b c*, and in the same direction, the angle A B C is equal to *a b c;* and in like manner it may be proved that the angle B C D is equal to *b c d*, and so on. Hence the two sections A B C D E and *a b c d e* are equal and similar.

(80) Cor. 1.—Hence all sections of a prism parallel to its bases are equal and similar to its bases.

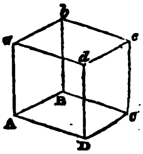
(81) Cor. 2.—All sections of a parallelopiped parallel to any face are parallelograms equal and similar to that face.

PROPOSITION VI.

(82) In every parallelopiped the opposite angles are included by plane angles which are equal each to each, but not similarly placed, so that the solid angles do not admit of being placed with their edges mutually coincident.

Since A *a* and *a b* are parallel to D *d* and *d c*, and in the same direction the angles A *a b* and D *d c* are equal. But the angle D *d c* is equal to the opposite angle D C *c*, therefore A *a b* is equal to D C *c*. In the same manner it may be proved that the angle A *a d* is equal to *c* C B and *b a d* to B C D.

But if the point C be conceived to be placed at *a*, and the edge C *c* upon *a* A, and C B upon *a d*, the edge C D will not coincide with *a b*, but will extend in the opposite direction from the vertex *a*; and in the same manner whatever pair of edges of the angle D be placed in coincidence with the pair of corresponding edges of B, the remaining edges will be found not to coincide.



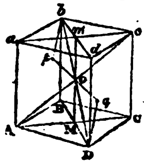
(83) DEF.—Two solid angles contained by plane angles which are equal each to each, but which do not admit of coincidence, are said to be *symmetrically equal.*

PROPOSITION VII.

(84) Of four parallel edges of a parallelopiped the diagonal planes which pass through each pair of them intersect, each dividing the other into two equal parallelograms, and the diagonals of each bisecting and being bisected by their common intersection.

Let A *a c* C be the diagonal plane through the opposite edges A *a* and C *c*, and D *d b* B that through the opposite edges D *d* and B *b*, and let *m* M be the intersection of these planes which will be equal and parallel to the edges A *a*, B *b*, &c.

The diagonals *a c* and *b d* of the face *a b c d* bisect each other at *m*, and in like manner A C and



B D bisect each other at M. Hence it is evident that the parallelogram A *m* is equal to the parallelogram C *m*, and in like manner B *m* is equal to D *m*.

Also since *a c* is bisected at *m*, and *m* O is parallel to *a* A, A *c* must be bisected at O. And since *m* O is half of *a* A, it is also half of *m* M which is equal to *a* A, and therefore *m* M is bisected at O. By similar reasoning it may be pròved that *b* D is bisected at O.

In this way it may be proved that every diagonal of the parallelopiped passes through O and is bisected at that point.

(85) Cor.—Every right line whatever drawn through the point O, and terminated in the faces of the parallelopiped, is bisected at O. For let *p q* be such a line terminated in the faces A *b* and D *c*; and draw *q* D and *p b*. The lines *p b* and *q* D are parallel, being the intersections of the plane of the lines *p q* and D *b* with the parallel planes D *c* and A *b*. Hence the triangles *q* O D and *b* O *p* are mutually equiangular, and since *b* O is equal to O D, they are mutually equilateral. Therefore *p* O is equal to *q* O.

(86) Def.—Hence the point O is called the *centre* of the parallelopiped.

(87) Def.—The quantity of space included within the surface or surfaces of a solid figure is called its *volume*.

Thus when we speak of the volume of a solid we do not take into account its *figure*. If any solid figure were divided in any manner into parts, and these parts changed in their arrangement so as to form another and different solid figure, still the *volume* would remain the same although the *shape* be altered.

Proposition VIII.

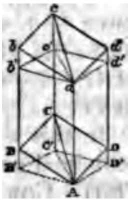
(88) The two prisms into which a diagonal plane divides a parallelopiped are equal in volume.

Let A B C D, *a b c d*, be the bases of the parallelopiped, and A *c* a diagonal plane. Through *a* and A draw planes at right angles, to the sides of the given parallelopiped so as to form a rectangular parallelopiped, of which the bases are A B′ C′ D′, *a b′ c′ d′*, and the sides of which coincide with those of the given one.

Since C *c* and C′ *c′* are each equal to A *a* they are equal to each other. Taking away the common part C *c′* we have *c c′*

equal to C C′. In the same manner we may prove that *d d′* is equal to D D′, and *b b′* to B B′.

Now suppose the triangle *a d′ c′* brought down and placed on A D′ C′ which is in every respect equal to it (79). Then the point *d′* being placed on D′, and the line *d′ d′* coinciding with D′ D, and being equal to it, the point *d* will coincide with D, and in the same manner *c* will coincide with C. Hence the vertices of the figure *a d′ d c c′* will severally coincide with those of the figure A D′ D C C′, and the figures themselves will therefore coincide and are therefore equal. To each of them let the figure A *a c′ d′* C D be added, and it follows that the prisms A C D, *a c d* and A C′ D′, *a c′ d′* are equal in volume.



In exactly the same way it may be proved that the prisms A B C, *a b c*, and A B′ C′, *a b′ c′* are equal in volume.

The prism A B′ C′, *a b′ c′* is equal to the prism A C′ D′, *a c′ d′* in every respect; for the bases A B′ C′, C′ D′ A are equal and similar, and the prisms are right (74). Hence the prisms A B C, *a b c* and A C D, *a c d*, which are respectively equal to them, are equal to each other.

It follows therefore that the volume of a parallelopiped is bisected by each of its diagonal planes.

(89) COR.—The triangular prisms into which an oblique parallelopiped is divided by a diagonal plane, although equal in volume, and also equal respectively as to their faces and edges, have not that equality which admits of coincidence. The solids are in this case said to be *symmetrically equal.*

The triangular prisms into which a rectangular parallelopiped is divided by a diagonal plane are not only equal but admit of coincidence.

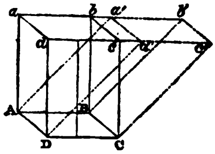
PROPOSITION IX.

(90) Two parallelopipeds on the same base and between the same parallel planes are equal in volume.

1°. Let their opposite bases be between the same parallels, *a b′* and *d c′*.

Let A B C D be the common base and *a b′ c′ d* the plane of the opposite bases parallel to A B C D.

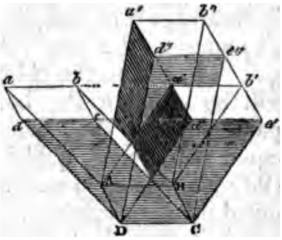
The triangular prisms A a', d' d and B b', c' c are equal and admit of coincidence; for if the face B b, c C be conceived to be placed upon A a, d D so as to coincide with it, it is evident that the lines C c' and B b' will coincide with the lines D d' and A a'. For the angles d D d' and c C c' are equal in consequence of the parallels; as also the angles a A a' and b B b' for the same reason. And since D d' is equal to C c', the point c' will coincide with d', and in the same manner b' will coincide with a'. Hence the two prisms will coincide, and are therefore equal.



Now if these equal prisms be successively taken from the solid A a b' c', the remainders will be equal, but these remainders are the parallelopipeds.

2°. Let their opposite bases be not placed between the same parallel lines.

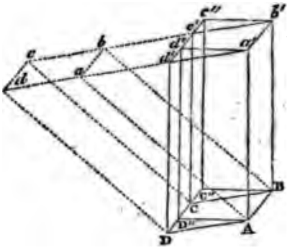
The opposite bases a b c d, a'' b'' c'' d'' must be equal in every respect since each is equal to the common base A B C D. If the sides of these opposite bases which are not parallel be produced until they intersect, a third parallelogram a' b' c' d' will be formed in the same plane with the upper bases, and equal to each of them in all respects and to the common base A B C D. The vertices a' b' c' d' being respectively joined by the right lines with A B C D, a third parallelopiped will be formed which will have a common base A B C D with each of the given ones, and which will have its opposite base between the same parallels as that of each of the given parallelopipeds. Hence this third parallelopiped is equal in volume (Part 1°.) to each of the given parallelopipeds, and they are therefore equal to each other in volume.



(91) Cor.—It is evident from this proposition that if two parallelopipeds have bases which are equal in all respects, and equal altitudes, they will have equal volumes. For if their bases be placed one upon the other, so that the solids shall lie at the same side of them, the opposite bases will be in the same plane parallel to that of the coincident bases.

Proposition X.

(92) An oblique parallelopiped is equal in volume to a rectangular one having a base of the same magnitude and an equal altitude.



On the base A B C D of the oblique parallelopiped construct another parallelopiped in the same altitude and whose sides will be perpendicular to the base. The volume of this will be equal to that of the oblique parallelopiped (90). If the base be a rectangle this latter parallelopiped will be rectangular, and the proposition is proved. But if the base be an oblique parallelogram take one of the perpendicular faces A B *b' a'* as base, and on it construct a rectangular parallelopiped between the same parallel planes as the last. It will be equal in volume to the latter and therefore to the oblique parallelopiped, and the base A B C D will be equal to the base A B C'' D''. Hence the rectangular parallelopiped thus constructed on an equal base, and with an equal altitude, is equal in volume to the oblique parallelopiped.

(93) Schol.—Hence it appears that the volume of a parallelopiped depends solely on the magnitudes of its base and altitude, and is independent of the obliquity of its edges or the figure of its faces.

Proposition XI.

(94) Rectangular parallelopipeds which have the same base are as their altitudes.

This is proved in exactly the same manner as the first proposition of the sixth book of the Elements. If the altitudes of the parallelopipeds be increased until they become equimultiples of their first magnitudes, the parallelopipeds themselves will be so increased as to become the same multiples of their first magnitudes. After this the reasoning is precisely the same here as in Prop. I., Book VI., the altitudes of the parallelopipeds taking the places of the bases of the triangles, and the volumes of the parallelopipeds taking the places of the areas of the triangles.

(95) Cor.—The same is true of rectangular parallelopipeds having bases in all respects equal, since by supposing them placed one upon the other they will have the same base.

Proposition XII.

(96) Rectangular parallelopipeds having the same altitude are as their bases.

Let A be the common altitude of the parallelopipeds, and let b c be the sides of one base and b' c' of the other. Let a third rectangular parallelopiped be constructed with the same altitude A, and with a base whose sides are b b'. Let the two proposed parallelopipeds be called A b c and A b' c' and the constructed one A b b'.

In the parallelopipeds A b c and A b b', the faces whose sides are A b are equal and may be considered as the bases, in which case c and b' will be the altitudes. Hence by (95)

$$\mathrm{A}bc : \mathrm{A}bb' :: c : b'.$$

In the same way in the parallelopipeds Abb' and A$b'c'$, the faces whose sides are Ab' are equal, and may be taken as the bases, in which case b and c' will be the altitudes, and we have (95)

$$\mathrm{A}bb' : \mathrm{A}b'c' :: b : c'.$$

Hence A b c is to Ab' c' in a ratio compounded of $c : b'$ and $b : c'$, or as the rectangle $b \times c : b' \times c'$, that is,

$$\mathrm{A}bc : \mathrm{A}b'c' :: b \times c : b' \times c'.$$

Hence the rectangular parallelopipeds having the altitude A and the bases $b \times c$ and $b' \times c'$ are as those bases.

(97) Cor.—The same will be true when the altitudes are equal, since by placing the bases in the same plane they will have the same altitude.

Proposition XIII.

(98) Rectangular parallelopipeds in general are in a ratio compounded of the ratios of their bases and altitudes.

Let A and A′ be the altitudes, and B and B′ the bases, and let a third parallelopiped be constructed with the altitude A and the base B′, and let the three solids be called A B, A′ B′, and A B′.

By (97) we have $\mathrm{A\,B} : \mathrm{A\,B'} :: \mathrm{B} : \mathrm{B'}$,
and by (95) $\mathrm{A\,B'} : \mathrm{A'B'} :: \mathrm{A} : \mathrm{A'}$.

But A B : A′B′ in a ratio compounded of A B : A B′, and A B′ : A′ B′, or of B : B′ and A : A′, that is, the parallelopipeds are in a ratio compounded of the ratios of their bases and altitudes.

Proposition XIV.

(99) Rectangular parallelopipeds are in a ratio compounded of the ratios of three conterminous edges.

Let $a\ b\ c$ be the three conterminous edges of one, and $a'\ b'\ c'$ those of the other. If a and a' be considered as the altitudes and $b \times c$ and $b' \times c'$ as the bases, the solids are in a ratio compounded of $a : a'$ and $b \times c : b' \times c'$. But the ratio $b \times c : b' \times c'$ is compounded of the ratios $b : c$ and $b' : c'$. Hence the parallelopipeds are in a ratio compounded of $a : a'$, $b : b$, and $c : c$., that is, of their three conterminous edges.

Proposition XV.

(100) If three conterminous edges of a rectangular parallelopiped be expressed by numbers, the product of these three numbers will express its volume, the units of such product being the cube of one of the parts into which the edges are supposed to be divided when numerically expressed.

When a line is expressed by a number, as 10, it is supposed to consist of ten equal parts of some conventional denomination, as inches, feet, yards, &c. When several lines entering the same computation are expressed by numbers, their parts are commonly taken to be equal. Thus if two lines be expressed by 10 and 8, we do not in general in the same computation consider one as 10 *inches* and the other 8 *feet*. They are either 10 feet and 8 feet, or 10 inches and 8 inches.

Let us suppose the three conterminous edges to be divided into inches, and let a, b, and c represent the number of inches in each of them.

A parallelopiped whose base is a square inch, and whose altitude is as many inches as there are units in a, has as many cubic inches in its volume as there are units in a. This is evident, since it is nothing more than a pillar of cubic inches laid one over another.

Hence a parallelopiped whose base is 2 or 3 square inches, and whose altitude is as many inches as there are units in a, has 2 or 3 times as many cubic inches in its volume as there are units in a, and in general if B be the number of square inches in the base $a \times B$ will be the number of cubic inches in the volume.

Hence it appears that the volume of a rectangular parallelopiped is represented numerically by multiplying together the numbers which express its base and altitude.

But the number of square inches in the base B is found by multiplying the number of inches in one side of the base by the number of inches in the other side. Numbers of inches in the sides of the base being b and c, the number of square inches in the base is $b \times c$. And since the number of cubic inches in the volume is $a \times B$, this number becomes $a \times b \times c$ when $b \times c$ is substituted for B.

It is evident that the same reasoning will be applicable if the edges be supposed to be divided into parts of any other denomination.

PROPOSITION XVI.

(101) The volume of a prism is expressed numerically by the product of the numbers which express its base and altitude.

If the prism be a rectangular parallelopiped, the proposition has been proved in (100).

If it be an oblique parallelopiped its volume is equal to that of a rectangular one having an equal base and altitude (92), and therefore it is expressed by the same product.

If it be a triangular prism it may be considered as the half of a parallelopiped cut off by a diagonal plane (88); and as its base is half, also the product of its base and altitude is half that of the base and altitude of the parallelopiped; and therefore the product represents the volume, which is also half of that of the parallelopiped.

If the prism have a polygonal base it may be divided into triangular prisms by diagonal planes (77). The volume of each of these is expressed by the product of the altitude and base, and therefore the volume of the whole will be expressed by the product of the altitude and the sum of the bases, that is, the product of the altitude and the base of the polygonal prism.

(102) COR. 1.—Hence prisms in general are in a ratio compounded of their bases and altitudes. Also when they have the same base they are as their altitudes, and when they have the same altitude they are as their bases.

(103) COR. 2.—Prisms are equal when their bases and altitudes are reciprocally proportional, and *vice versâ*, whatever be the figures of their bases or the inclinations of their sides.

(104) Def.—Prisms are said to be *similar* when their bases are similar rectilinear figures, and their sides are proportional to homologous sides of their bases, and similarly inclined to those bases.

Proposition XVII.

(105) Similar prisms are in the triplicate ratio of their homologous edges.

For they are in a ratio compounded of the ratios of their bases and altitudes (102). But their bases are in the duplicate ratio of their homologous sides which are homologous edges of the prisms, and since the sides are equally inclined to the bases, the altitudes are as the sides; but the sides of the prism are proportional to homologous sides of the bases (104), therefore the altitudes are as homologous sides of the bases, or as homologous edges of the prisms. Hence the prisms are in a ratio compounded of the duplicate and simple ratios of their homologous edges, that is, in the triplicate ratio of their homologous edges.

(106) Cor. 1.—Since *cubes* are similar prisms, it follows that cubes are in the triplicate ratio of their edges.

(107) Cor. 2.—Similar prisms are as the cubes of their homologous edges.

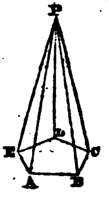
(108) Cor. 3.—If the first two of four lines in continued proportion be homologous sides of similar prisms, the volumes of the prisms will be as the first line to the fourth.

(109) Cor. 4.—In order to construct a prism similar to a given one and to which the given prism shall have a given ratio, it will be necessary to find two mean proportionals between an edge of the given prism and a line to which that edge bears the given ratio. If on the first of these two means a prism be constructed similar and similarly placed with the given prism, it will be that which is required (108).

(110) Cor. 5.—If the given prism be a cube, and the given ratio be 1 : 2, the preceding corollary becomes the celebrated Delian problem, *the duplication of the cube*, for an account of which see Elements (586).

(111) Schol.—It will hereafter appear that the preceding corollaries may be extended to all similar solids.

(112) Def.—A *pyramid* is a solid having several triangular faces which have the same vertex P, and whose bases are the sides of the remaining face, which may be any rectilinear figure A B C D E, and which is called the *base* of the pyramid, the common vertex P of the triangular faces being called the *vertex* of the pyramid



The areas of its triangular faces form the *lateral* or *convex surface* of the pyramid.

The sides P A, P B, &c. of the triangular faces are called the *sides* of the pyramid.

(113) Def.—The *altitude* of a pyramid is the perpendicular distance of the vertex from the plane of the base.

(114) Def.—A *pyramid* is denominated *triangular*, *quadrangular* or *polygonal*, according as the base is *triangular*, *quadrangular* or *polygonal.*

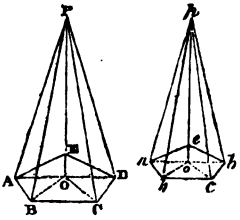
(115) Def.—A regular pyramid is one whose base is a regular polygon, and whose vertex is so placed that a perpendicular drawn from it to the plane of the base will meet that plane in the centre of the base. This perpendicular is in this case called the *axis* of the pyramid.

(116) Def.—*Similar pyramids* are those which have similar bases, altitudes proportional to the homologous sides of those bases and vertices so placed that perpendiculars drawn from them to the planes of the bases will meet these bases at points similarly placed.

Proposition XVIII.

(117) The sides of similar pyramids, which are conterminous with homologous sides of their bases, are proportional to homologous sides of the bases, and are similarly inclined to the planes of the bases.

For let P O and *p o* be the perpendiculars from the vertices of the pyramids upon the planes of the bases, and in those planes let lines be drawn from the points O and *o* to the several vertices of the angles A, B, &c. *a*, *b*, &c. By (116) it follows that the several lines O P, O A, O B, O C, &c. are respectively to *o p*, *o a*, *o b*, *o c*, &c. as any pair of homologous sides of the bases. Hence in the triangles P O A and *p o a*, P O : O A : : *p o* : *o a*, and the angles O and *o* being right, the triangles P O A and *p o a* are similar. Therefore P A : *pa* : : P O : *p o*, that is (116) as A B : *a b*, or as any pair of homologous sides. Also the angles A P O and *a p o* are equal, and therefore the sides A P and *a p* are equally inclined to the bases. In the same manner it may be proved, that each pair of corre-



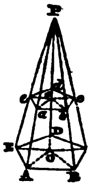
sponding sides of the pyramid are proportional to each pair of homologous sides of the base, and similarly inclined to the base.

(118) Cor.—The planes of the triangles A P O, B P O, C P O, &c. *a p o*, *b p o*, *c p o*, &c. divide the pyramids into a series of triangular pyramids which are similar each to each.

Proposition XIX.

(119) If a pyramid be intersected by parallel planes, the pyramids cut off by those planes will be similar.

Let A B C D E and *a b c d e* be the parallel sections, and from the vertex P let the perpendicular P *o* O be drawn to the parallel planes.



Since the parallel planes A B C D E and *a b c d e* are intersected by the plane A P B, the intersections A B and *a b* are parallel, and in the same manner it may be proved that B C is parallel to *b c*, C D to *c d* and so on. Hence (44) it follows that the sections A B C D E and *a b c d e* are mutually equiangular. But also the triangles A P B and *a* P *b* are similar; therefore A B : *a b* : : B P : *b* P, and in like manner B C : *b c* : : B P : *b* P. Hence A B : *a b* : : B P : *b* P; and, by the same reasoning, each pair of corresponding sides of the sections may be proved proportional.

The triangles A P O and *a* P *o* are evidently similar, therefore A O : *a o* : : P O : P *o* : : A P : *a* P : : A B : *a b;* and the same being applicable to the lines B O, *b o* and C O, *c o*, &c., it follows that the points O and *o* are similarly placed in the sections, and that the altitudes P O, P *o* are proportional to the homologous sides.

(120) Cor. 1.—Hence all sections of a pyramid parallel to the base are similar to the base, and the pyramids cut off by such sections are similar to the original pyramid.

(121) Cor. 2.—The sections A B C D E and *a b c d e* are in the duplicate ratio of the perpendiculars P O and P *o;* for these perpendiculars are as the homologous sides.

(122) Cor. 3.—In any two pyramids sections parallel to their bases which divide their altitudes proportionally are as their bases. For the bases are to the sections in the duplicate ratio of the altitudes to the segments of the altitudes between them and the vertices. But these segments are as the altitudes themselves, by *hypothesis*. Therefore the bases are as the sections.

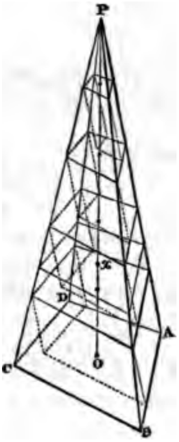
(123) Cor. 4.—If the bases of two pyramids be equal, sections

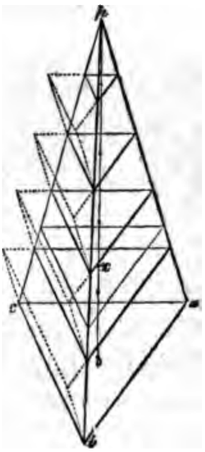
dividing the altitude proportionally are equal, and if both the altitudes and bases be equal, all sections equally distant from the vertices are equal in magnitude.

PROPOSITION XX.

(124) Pyramids which have equal altitudes and bases of equal areas, have equal volumes.

If they be not equal in volume let A B C D, &c. be the base of the greater, and let O x be the altitude of a prism whose base is A B C D, and whose volume is equal to the difference of the two pyramids.

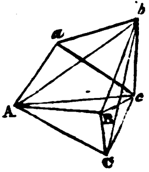




Let the equal altitudes P O, $p\,o$ be divided into such a number of equal parts that each part shall be less than O x; and let the pyramids be cut by planes parallel to their bases and passing through each of the points of division of the equal altitudes. The several sections of the pyramids by these planes will be equal each to each, since their bases and altitudes are equal (123). If prisms be constructed with these sections severally as bases and with the equal parts of the altitudes as altitudes, these prisms will be equal each to each (103). Let these prisms be constructed in the one pyramid *below* the sections respectively, and so as to lie *within* the pyramid, and in the other *above* the sections, and so as to lie partly *without* it. The sum of the prisms in the pyramid P is less than the pyramid, and the sum of those in the pyramid p is greater than the pyramid. Hence the difference between the sums of the prisms is greater than the difference between the pyramids. But the difference between the sums of the prisms is the prism whose base is A B C D, and whose altitude is one of the equal parts into which the altitude P O is divided, and the difference of the pyramids is a prism with the same base and the altitude O x. Hence O x is less than one of the parts into which the altitude is divided, but, by hypothesis, it is greater. Therefore the volumes of the pyramids cannot be unequal.

PROPOSITION XXI.

(125) The volume, a triangular pyramid, is one-third of that of a prism having an equal base and altitude.



Let A B C be the base of the pyramid and *c* its vertex. Through B and A draw B *b* and A *a* parallel to C *c* and equal to it, and draw the lines *a b c*. The prism A B C *a b c* is on the same base and has the same altitude as the pyramid. The solid which has thus been added to the pyramid to complete the prism, is a quadrangular pyramid whose base is A *a b* B, and whose vertex is *c*. Draw A *b*, and through this line and the vertex *c* draw the plane A *b c*. This plane divides the quadrangular pyramid into two triangular pyramids having a common vertex *c* and equal bases A *a b* and A B *b*. These pyramids are therefore equal (124). But the pyramid A *a b c* has a base *b c a* equal to A B C, and an altitude equal to that of the pyramid whose base is A B C and whose vertex is *c*. Hence these are equal (124), and therefore the three triangular pyramids which form the prism are equal in volume. Hence the triangular pyramid A B C *c* is a third of the volume of a triangular prism having the same base and altitude, and therefore also (103) a third of one which has an equal base and altitude.

PROPOSITION XXII.

(126) The volume of every pyramid is one-third of the volume of a prism having an equal base and altitude.

For every pyramid is equal to a triangular pyramid with an equal base and altitude (124), and every prism is equal to a triangular prism with an equal base and altitude (103). Hence the proposition is evident by (125).

(127) COR. 1.—Hence the volume of a prism is found numerically by multiplying its altitude by its base, and taking one-third of the product.

(128) COR. 2.—Pyramids are in a ratio compounded of the ratios of their bases and altitudes.

(129) COR. 3.—Pyramids having equal bases are as their altitudes, and those which have equal altitudes are as their bases.

(130) COR. 4.—Equal pyramids reciprocate their bases and altitudes, and *vice versâ*.

(131) Cor. 5.—Similar pyramids are as the cubes of their homologous edges, or in the triplicate ratio of these lines.

(132) Cor. 6.—Every solid figure which is bounded by planes can obviously be divided into triangular pyramids by planes drawn through any one of its vertices. Hence the volume of every solid may be found by determining the volumes of all the component pyramids by (126).

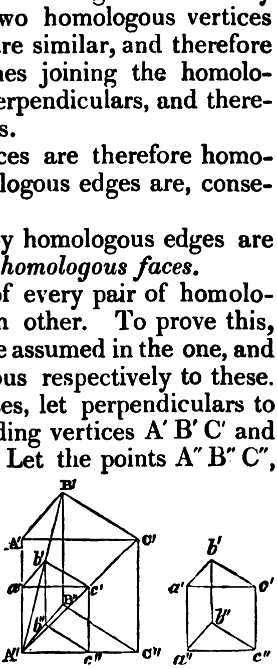
(133) Def.—Similar polyedrons are solids constructed on similar rectilinear figures as bases, and having the same number of vertices similarly placed with respect to those bases. That is, so placed that if perpendiculars be drawn from them severally these perpendiculars will meet the planes of the bases at points similarly placed, and that the perpendiculars themselves shall be each to each as homologous sides of the bases.

From this definition it easily appears, that the right lines joining any homologous vertices are proportional to homologous sides of the bases. For the plane quadrilateral figures formed by the perpendiculars on the base from two homologous vertices and right lines joining their extremities are similar, and therefore their sides proportional. Hence the lines joining the homologous vertices are as two corresponding perpendiculars, and therefore as two homologous sides of the bases.

These lines joining homologous vertices are therefore homologous edges of the solids, and all homologous edges are, consequently, proportional.

(134) Hence also the faces bounded by homologous edges are similar each to each, and may be called *homologous faces.*

(135) It also follows that the planes of every pair of homologous faces are equally inclined to each other. To prove this, let any two faces $ABCD$, $A'B'C'D'$* be assumed in the one, and also two faces $abcd$, $a'b'c'd'$ homologous respectively to these. Considering $ABCD$, $abcd$ as the bases, let perpendiculars to them be drawn from any three corresponding vertices $A'B'C'$ and $a'b'c'$ of the other homologous faces. Let the points $A''B''C''$, $a''b''c''$, where the perpendiculars meet the planes of the bases, be joined by right lines, and let planes be conceived to be drawn so as to form the solids $A'B'C'A''B''C''$, $a'b'c'a''b''c''$.



From the definition of similar solids and its consequences, it follows that the triangles $A''B''C''$ and $a''b''c''$ are similar, and that their homologous sides are as the perpendiculars respectively, or as any homologous edges of the given solids. Let the point a'' be conceived to be placed on A'', and the sides $a''b''$ and $a''c''$ upon $A''B''$ and $A''C''$ respectively.

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* These faces are not represented in the cut.

Since $c''c'$ and $C''C'$ are parallel, they are in the same plane with each other and the line $A''C''$ which intersects them. We have $A''c'' : c''c' :: A''C'' : C''C'$ and the angles $A''c''c'$ and $A''C''C'$ equal, being right. Therefore the triangles $A''c''c'$ and $A''C''C'$ are similar; and the points $A''c'C'$ are in the same straight line, and $A''c' : A''C'' :: A''c'' : A''C''$. Also we have $A''a' : A''A' :: A''c'' : A''C'' :: A''c' : A''C'$. Hence the triangles $a'A''c'$, $A'A''C'$ are similar, and the lines $a'c'$, $A'C'$ are parallel. In the same manner it may be proved that the lines $a'b'$ and $A'B'$ are parallel, and therefore the planes $b'a'c'$ and $B'A'C'$ are parallel, and are therefore equally inclined to the plane $B''A''C''$.

In this way we prove, that if the bases of similar polyedrons be placed one upon the other in such a manner that the homologous sides of those bases will be parallel, then the planes of every pair of homologous faces will be parallel, and they will be respectively similarly inclined to the bases and to each other, and in every respect similarly placed.

(136) By similar reasoning, the student will find no difficulty in proving that the diagonal planes drawn through any three homologous edges of two similar polyedrons, cut off similar triangular pyramids, and as such similar pyramids may be continually cut off until the polyedrons themselves are reduced to two similar triangular pyramids, it follows that "similar polyedrons may be divided into triangular pyramids equal in number and similar each to each."

PROPOSITION XXIII.

(137) The volumes of similar polyedrons are as the cubes of their homologous edges.

For the several pyramids into which they may be resolved being similar each to each, are as the cubes of their homologous edges (131), that is, as the cubes of homologous edges of the polyedrons. The pyramids being each to each in this ratio, their sums will be in the same ratio; but these sums are the polyedrons.

(138) COR. 1.—Hence if four right lines be in continued proportion, and the first two be homologous edges of similar polyedrons, the volumes of these solids will be as the first line to the fourth.

(139) Since the faces of similar polyedrons are similar each to each, their magnitudes are as the squares of their homologous edges, and therefore the sums of all the faces are in the same

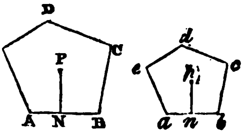
ratio. Hence the surfaces of similar polyedrons are as the squares of their homologous edges.

PROPOSITION XXIV.

(140) To construct a solid similar to a given solid, and such that the volume of the given solid shall bear to the volume of the constructed one a given ratio.

The geometrical solution of this problem would require the solution of the problem " to find two mean proportionals," and therefore it cannot be solved by a strictly geometrical process. If we suppose, however, the two mean proportionals to be found, the problem admits of strict solution. That is, we can construct upon a given right line as an edge a solid similar and similarly placed with a given solid.

Let a line be assumed to which one of the edges of the given solid bears the given ratio, and between this edge and the line so assumed conceive two mean proportionals to be found. The first of these mean proportionals must be an edge of the solid to be constructed, homologous to that edge of the given solid which is taken as the antecedent of the given ratio.



Let A B be the edge of the given solid which is taken as antecedent of the given ratio, and let A B C D E be one of the faces of the solid, of which this edge is a side. Let ab be the first mean. On ab construct a rectilinear figure $abcde$ similar and similarly placed with A B C D E on A B. From all the vertices of the solid which stands on A B C D E, let perpendiculars be drawn upon the plane A B C D E, and let P be the foot of one of these perpendiculars. Draw P N perpendicular to A B, and let ab be cut at n similarly to A B at N, and draw the perpendicular pn such that $pn : PN :: ab : AB$. Then p and P are homologous points in the similar figures A B C D E and $abcde$.

From the point p let a perpendicular to the plane $abcde$ be drawn, such that it shall have to the perpendicular, whose foot is P, the ratio $ab : AB$. The end of the perpendicular thus drawn will determine that vertex of the solid to be constructed, which is homologous to the vertex of the given solid from which the perpendicular P is drawn.

In the same manner each of the other vertices of the required solid is determined.

The similarity of the solid thus constructed with the given solid will easily appear from the results of (133) *et seq.*

(141) It should be observed, that in this book we confine our investigations to what are called *convex polyedrons*, or those which have no solid angles whose vertices are presented *inwards*. The most distinct test of a convex figure, whether plane or solid, is, that its perimeter or surface can be only intersected by a right line in two points.

(142) Def.—The solid figure included between two parallel planes intersecting a pyramid is called a *truncated pyramid*.

Proposition XXV.

(143) The volume of a truncated pyramid is equal to the sum of the volumes of three pyramids having the same altitude as the truncated pyramid, and whose bases are those of the truncated pyramid and a mean proportional between them.

This proportion is most simply investigated by the aid of algebraical notation. Let a, a' be homologous sides of the similar bases (119) of the truncated pyramid. The squares of these lines are as the areas of the bases, or the square of each line bears the same ratio to the area of the base of which it is a side. Hence if m be such a number that $m a^2$ is equal to the one base, $m a'^2$ will be equal to the other. Now, suppose the sides of the truncated pyramid produced till they meet so as to complete the pyramid, and let a perpendicular be drawn from the vertex to the parallel bases of the truncated pyramid. Let the length of this perpendicular drawn to the greater base be h, and to the lesser h'. The volume of the pyramid having the greater base $m a^2$ is then $\frac{1}{3} h m a^2$ (127), and that of the lesser $\frac{1}{3} h' m a'^2$, and therefore the volume of the truncated pyramid is $\frac{1}{3} h m a^2 - \frac{1}{3} h' m a'^2 = \frac{1}{3} m (h a^2 - h' a'^2)$. But (122) $h : h' :: a : a'$, and hence we easily infer

$$\frac{h}{h-h'} = \frac{a}{a-a'}, \quad \frac{h'}{h-h'} = \frac{a'}{a-a'}.$$

Now let V be the volume of the truncated pyramid

$$\tfrac{1}{3} m (h a^2 - h' a'^2) = \mathrm{V},$$

divide these equals by $h - h'$, and we find

$$\tfrac{1}{3} m \left(\frac{h}{h-h'} a^2 - \frac{h'}{h-h'} a'^2\right) = \frac{\mathrm{V}}{h-h'},$$

or $$\tfrac{1}{3}m\left(\frac{a}{a-a'}a^2 - \frac{a'}{a-a'}a'^2\right) = \frac{V}{h-h'},$$

or $$\tfrac{1}{3}m \cdot \frac{a^3 - a'^3}{a-a'} = \frac{V}{h-h'}.$$

But $h - h'$ is the altitude of the truncated pyramid. Call this H; and if $a^3 - a'^3$ be divided by $a - a'$, the quote is $a^2 + a\,a' + a'^2$. Hence we have

$$\tfrac{1}{3}m\,(a^2 + a\,a' + a'^2) = \frac{V}{H},$$

or $$\tfrac{1}{3}\,H\,m\,a^2 + \tfrac{1}{3}\,H\,m\,a'^2 + \tfrac{1}{3}\,H\,m\,a\,a' = V.$$

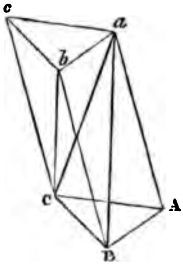
Now $m\,a^2$ is the greater base, and $\frac{1}{3}H\,m\,a^2$ is the volume of a pyramid whose base is $m\,a^2$ and whose height is H, or that of the truncated pyramid. In like manner $\frac{1}{3}H\,m\,a'^2$ is the volume of a pyramid whose base is the lesser base $m\,a'^2$ of the truncated pyramid, and whose height is H, that of the truncated pyramid; and, lastly, $\frac{1}{3}H\,m\,a\,a'$ is the volume of a pyramid whose base is $m\,a\,a'$ (a mean proportional between $m\,a^2$ and $m\,a'^2$) and whose altitude is H, that of the truncated pyramid.

Proposition XXVI.

(144) The volume of a solid included by two planes intersecting the sides of a triangular prism, is equal to the sum of the volumes of three pyramids whose common base is either of the sections, and whose vertices are the vertices of the angles of the other section.

If the two intersecting planes be parallel, the solid is a triangular prism, and the three pyramids are equal, and hence the proposition becomes identical with (126).

If the planes be not parallel, let the triangular sections be A B C and *a b c*. Draw the plane B C *a*. This divides the solid into the pyramid A B C *a*, and the solid *a b c* B C. Draw the plane *a b* C. This divides the latter solid into two pyramids, whose bases are *b c* C and B C *b*, and of which *a* is the common vertex. The pyramid whose base is B C *b* and vertex *a* is equal to that with the same base B C *b* and vertex A; because *a* A is parallel to the plane B C *b*, and therefore the two pyramids have equal altitudes. But the pyramid A C *a* B is that which is on the base A B C, and has the vertex *a*. Hence, in like manner, the pyramid C *a c b* is equal



to C *b* *c* A, for they have the same base C *b* *c* and equal altitudes, since A *a* is parallel to that base. But the pyramid A *b* *c* C is equal to the pyramid A B *c* C, for they have the same base A *c* C and equal altitudes, since B *b* is parallel to that base. Hence it appears that the three pyramids A B C *a*, A C *c* *b*, and B *a* C *b*, into which the solid is resolved by the planes B *a* C and *a* *b* C through the vertex *a*, are respectively equal to three pyramids, of which A B C is the base, and whose vertices are at the points *a*, *b*, *c*.

BOOK III.

Of the Regular Solids.

(145) Def.—A regular solid is one whose faces are equal regular polygons, and whose solid angles are equal.

Proposition XXVII.

(146) There cannot be more than five regular solids.

1°. Let the faces be equilateral triangles. A solid angle may be formed by three, four, or five plane angles, each of which is two thirds of a right angle. But six or more angles of this magnitude would be equal to or greater than four right angles; and, consequently, could not form a solid angle. The number of regular solids, therefore, whose faces are triangular cannot exceed three.

2°. Let the faces be squares. A solid angle may be formed of three right angles, but not of a greater number. Wherefore there is but one regular solid with square faces.

3°. Let the faces be pentagons. A solid angle may be formed of three angles of a regular pentagon; for the magnitude of one is six fifths of a right angle, and therefore the aggregate magnitude of three such angles is $\frac{18}{5}$ of a right angle, or three right angles and three fifths, which is less than four right angles. But four or more such angles will be evidently greater than four right angles, and therefore cannot form a solid angle. Hence there cannot be more than one regular solid with pentagonal faces.

4°. If the faces were hexagons the angles would be four thirds of a right angle, and three such angles would be equal to four right angles, and therefore could not form a solid angle; and it is evident that no greater number of such angles than three could form a solid angle. If the faces were polygons with more than six sides their angles would be greater than those of a regular hexagon, and similar observations would be applicable. Hence no regular solid can have faces with more sides than five. Hence we infer

1°. That there cannot be more than five regular solids.

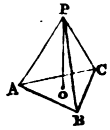
2°. That three of these have triangular faces, one square faces, and the remaining one pentagonal faces.

3°. That the solid angles of the three regular solids with triangular faces, are formed of three, four, and five plane angles, and that the solid angles of the others are formed of three plane angles.

Proposition XXVIII.

(147) To construct a regular solid, whose faces are triangular, and whose solid angles are contained by three plane angles.

From the centre O of an equilateral triangle A C B draw a perpendicular O P to its plane, and on this perpendicular from the points A, B, C inflect lines A P, B P, C P equal to the side of the equilateral triangle. It is evident that these lines will meet the perpendicular in the same point P. A pyramid will thus be formed on the base A B C, having its vertex at P, the four faces of which will be equal equilateral triangles. This is therefore the regular solid required.



This solid is therefore the *regular tetraedron*.

(148) Cor. 1.—It is evident that each pair of faces of the tetraedron form the same angle. The magnitude of this angle may be easily calculated by spherical geometry and trigonometry, (160). It is, in fact, the angle of an equilateral spherical triangle, whose side is 60°. The calculation is as follows. Let the angle be A, we have then

$$\cos a - \cos A \sin b \sin c - \cos b \cos c = 0.$$

See my Trigonometry (181), first formula of [1].

In this let a, b, c be supposed to be each $= 60°$. Since $\cos 60° = \frac{1}{2}$ and $\sin 60° = \frac{\sqrt{3}}{2}$ the formula becomes

$$\frac{1}{2} - \cos A \,.\, \frac{3}{4} - \frac{1}{4} = 0, \because \cos A = \frac{1}{3}.$$

Hence the angle under each pair of faces is one whose cosine is

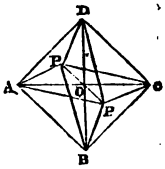
$\frac{1}{3}$, or it is the greater acute angle in a right angled triangle, one side of which is one third of the hypotenuse.

(149) Cor. 2.—The volume of a regular tetraedron may now be determined numerically. Let the edge be the unit. The radius O A of the circle which circumscribes the base will be $\frac{1}{\sqrt{3}}$. This line, the edge A P, (which is $= 1$,) and the perpendicular O P, form a right angled triangle. Hence the perpendicular is $\sqrt{1 - \frac{1}{3}} = \sqrt{\frac{2}{3}}$. The area of the base is $\frac{\sqrt{3}}{4}$; one third of the product of this and the perpendicular is then $\frac{\sqrt{3}}{4} \times \sqrt{\frac{2}{3}} \times \frac{1}{3} = \frac{1}{6 \, . \sqrt{2}}$, which expresses the proportion of the tetraedron to a cube constructed with an equal edge.

Proposition XXIX.

(150) To construct a regular solid with triangular faces, and whose solid angles are formed by four plane angles.

Construct a square A B C D, and through its centre O draw a perpendicular, producing it on both sides of the plane of the square. From the points A B C D inflect on this perpendicular, at both sides of the planes, lines equal to the side of the square. It is evident that those which are on the same side of the plane will meet the perpendicular at the same point. Let these points be P and P′. Two pyramids will thus be constructed on opposite sides of the square, and the lateral faces of each of these pyramids will be the equilateral triangles constructed on the sides of the square. These two pyramids united form a regular solid with eight triangular sides, and of which the square is a diagonal plane.



This solid is therefore the *regular octaedron.*

(151) Cor. 1.—To determine the inclination of the planes of each pair of intersecting faces, let them be considered as two sides of a solid angle of the pyramid whose base is the square. This solid angle is then formed by three plane angles, two of which are 60°, and the third a right angle. In the formula used in (148), let $a = 90°$, $b = c = 60°$, and it becomes

$$- \cos A \, . \frac{3}{4} - \frac{1}{4} = 0, \because \cos A = - \frac{1}{3}.$$

Hence A is an obtuse angle whose cosine is $\frac{1}{3}$, or it is the external angle to the greater obtuse angle in a right angled triangle, one side of which is a third of the hypotenuse.

(152) Cor. 2.—Hence it appears that the angle under the adjacent planes of the regular octaedron, is the supplement of the angle under the adjacent planes of the regular tetraedron.

(153) Cor. 3.—If three faces of the octraedron whose bases form the sides of the same face, (such as A D P, B C P, A P′ B,) be produced through those sides until they form a solid angle, the produced parts will form a regular tetraedron with the face through whose sides they are produced.

(154) Cor. 4.—Each pair of faces of the octaedron (A P B, D P′ C,) which are constructed on opposite sides, A B, D C of the square, and also on opposite sides of its plane are parallel. For the alternate angles which their planes form with that of the square are equal.

(155) Cor. 5.—If the planes of three faces which are terminated in the sides of any one face A B P be produced until they form a solid angle, and also until they meet the plane of the face D C P′ which is parallel to A B P produced, they will with it form a regular tetraedron circumscribing the octaedron. Each face of this tetraedron will be divided into four equal equilateral triangles, by the face of the octaedron by whose production it is formed.

Hence it follows that the whole surface of the tetraedron is equal to sixteen times one of the faces of the octaedron, and these to double the whole surface of the octaedron.

It appears, therefore, that if the four corners be cut from a regular tetraedron by planes through the points of bisection of every three conterminous edges, the remaining figure will be a regular octaedron. Since each pyramid which is thus cut off is similar to the whole, and the edges are in the proportion of 1 : 2, the volumes are as 1 : 8. Therefore each of the four pyramids is equal to an eighth of the original pyramid, and to a fourth of the octaedron which remains after their removal.

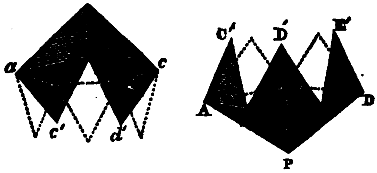
(156) Hence it appears, that the volume of a regular octaedron whose edge is 1, is half the volume of a regular tetraedron whose edge is 2. But by (149) the volume of a tetraedron whose edge is 1 is $\frac{1}{6\,.\,\sqrt{2}}$. And since a similar solid whose edge is 2 has eight times the volume (137), it follows that the volume of a tetraedron whose edge is 2 is $\frac{8}{6\,\sqrt{2}}$, or $\frac{2\sqrt{2}}{3}$. Hence the regular octae-

dron whose edge is 1 is $\frac{\sqrt{2}}{3}$. The volumes of an octaedron and cube constructed on the same edge are therefore as $\sqrt{2} : 3$.

Proposition XXX.

(157) To construct a regular solid with triangular faces and whose solid angles are formed by five plane angles.

Let a regular pentagon A B C D E be constructed, and through its centre let a perpendicular to its plane be drawn. From the points A, B, C, D, E, respectively, let right lines equal to the side of the pentagon be inflected on this perpendicular. Since the side of a regular pentagon is greater than the radius of its circumscribing circle, these lines will meet the perpendicular below the plane of the pentagon; and since the lines so inflected are equal, they will meet the perpendicular at the same point P so as to form a regular pentagonal pyramid. The solid angle P at the vertex of this pyramid will be then formed of five plane angles, each of which is two thirds of a right angle. Two of the plane angles which form each solid angle at the base of the pyramid, have evidently the same inclination as any two of the plane angles which form the solid angle P, being, in fact, the same planes. Hence the solid angles A, B, C, &c. at the base may be considered parts of solid angles equal to P formed by five plane angles, the part included by three of the plane angles being cut off by the plane angle of the base of the pyramid.



On each side of the base of the pyramid let an equilateral triangle be constructed, so that its plane shall be inclined to the adjacent lateral face of the pyramid at the same angle as any two of the adjacent lateral faces; that is, so that the angle under the planes A B C′ and A B P shall be equal to the angle under any two adjacent planes containing the angle P, and so that the same may be true of the planes B C D′ and B C P, C D E′ and C D P, &c.

Hence it follows, that at each of the vertices A, B, C, &c. of the base of the pyramid there are four angles each two thirds of a right angle, and which are united at the same inclinations as four of the angles which form the solid angle P. It follows, therefore, that the angle C′ B D′ included between the conterminous sides (B C′, B D′) of two equilateral triangles A B C′, C B D′, constructed upon conterminous sides of the pentagonal base must be an angle of an equilateral triangle, so placed that if its plane be supposed to be drawn it will complete the solid angle B, and render it equal to P. The same conclusion is obviously applicable to each of the other angular points of the base.

We have thus a figure formed having a solid angle at P formed of five angles of equilateral triangles, having ten equilateral triangular faces, and a serrated edge or boundary A C′ B D′ C E′, &c., the planes of the angles being so disposed that if the gaps C′ B D′, D′ C E′, &c. be filled up, solid angles will be formed at A, B, C, &c. equal to P.

Let another figure in every respect equal and similar to this be formed, the corresponding points being marked by the small letters *a*, *b*, *c*, ... *a′*, *b′*, *c′*, &c. Let the point *c′* be placed upon B, and the sides *c′ a*, *c′ b*, upon the equal sides B C′, B D′ of the equal angle C′ B D′. It is evident that the points *a* and *b* will coincide with C′ and D′ respectively. Thus the angle *a c′ b* inserted in C′ B D′ will complete the solid angle B, which will then be equal to P.

The plane of the angle D′ B C has been already proved to be inclined to that of D′ B C′ at the same angle as any two adjacent plane angles of P, and the same is true of the planes of the angles *a c′ b* and *c′ b d′*. Since then the plane *a c′ b* coincides with C′ B D′, and the planes *c′ b d′* and B D′ C are equally inclined to that plane, the plane *c′ b d′* must coincide with B D′ C. Since the line B D′ coincides with *c′ b*, and the angles B D′ C and *c′ b d′* are equal, and in the same plane, the point *d′* must coincide with C. In the same manner we may prove that the points *c*, *e′*, &c. coincide with E′ D, &c.; and we may prove that each of the solid angles at these points is equal to P, as we have already proved of the solid angle B.

Hence it appears, that by the union of the two shells formed of ten equilateral triangles in the manner already described, a regular solid with twenty triangular faces is formed.

This solid is called the *regular icosaedron.*

(158) Cor.—To determine the inclination of the planes of each pair of intersecting faces. The edges B A, B C, and B P form a solid angle, the two plane angles P B A and P B C being those whose inclination is sought. Applying the formula used in the former cases, let *a*, *b*, and *c* be the plane angles A B C, P B C, and P B A. Hence $a = 108°$ (the angle of a regular pentagon,) $b = c = 60°$. Hence,

$$\cos 108° - \cos A \,.\, \frac{3}{4} - \frac{1}{4} = 0,$$

$$\therefore \cos A = \frac{4 \cos 108° - 1}{3} = - \frac{4 \cos 18° + 1}{3}.$$

The value of A may hence be found by the aid of tables.

PROPOSITION XXXI.

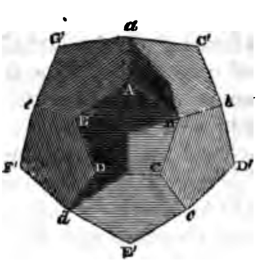
(159) To construct a regular solid with square faces.

This is, obviously, a rectangular parallelopiped whose base is a square, and whose altitude is equal to the side of the base.

The *regular hexaedron* is therefore the cube.

PROPOSITION XXXII.

(160) To construct a regular solid with pentagonal faces.



Let A B C D E be a regular pentagon. From the vertex A, draw the line A *a* equal to the side of the pentagon, and inclined to A B and A E at angles equal to the angle of the pentagon. The solid angle formed by the three lines which meet at that point is one of the angles of the required solid, formed by the three pentagonal angles *a* A B, *a* A E, and B A E. In the same manner, let the lines B *b*, C *c*, &c. be drawn from each of the angles of the pentagon forming solid angles of the same kind at the points B, C, D, &c. Let the pentagon, of which *a* A B *b* are three sides, be completed, and in the same manner let each of the other pentagons on the sides of the base A B C D E be completed. We shall thus have a shell with six regular and equal pentagonal faces, and a serrated edge, *a* C′ *b* D′ *c*, &c. The adjacent planes forming several pentagonal faces, are inclined each to each at the same angle; and it may be proved in the same manner as in (157), that if a plane be drawn through the angle C′ *b* D′, a solid angle will be formed at *b* equal to those at A, B, C, &c. As in (157), let another shell in every respect equal and similar to this be constructed, and let them be united at their serrated edges. It will follow by the reasoning used in the former case, that the several solid angles which will be formed at *a*, C′, *b*, D′, &c. will be equal to those at A, B, C, &c.

Hence by the union of those two shells with six pentagonal faces, a regular solid with twelve pentagonal faces is formed.

This solid is called the *regular dodecaedron*.

(161) COR.—To determine the inclination of the planes, we have

each solid angle formed by three plane angles of 108°. In the formula used in (148), let $a = b = c = 108°$, and it becomes

$$\cos 108° - \cos A \sin^2 108° - \cos^2 108^2 = 0,$$

$$\therefore \cos A = \frac{\cos 108° (1 - \cos 108°)}{1 - \cos^2 108°} = \frac{\cos 108°}{1 + \cos 108°},$$

$$\therefore \cos A = -\frac{\sin 18°}{1 - \sin 18°}.$$

But $\sin 18° = \frac{\sqrt{5} - 1}{4}$. (See Trigonometry (354).) Hence

$$\frac{\sin 18°}{1 - \sin 18°} = \frac{\sqrt{5} - 1}{4} \div \frac{5 - \sqrt{5}}{5}$$

$$= \frac{\sqrt{5} - 1}{5 - \sqrt{5}} = \frac{1}{\sqrt{5}},$$

$$\therefore \cos A = -\frac{1}{\sqrt{5}}.$$

Hence if a right angled triangle be constructed in which the square of one side is one fifth of the square of the hypotenuse, the angle opposite to the other side will be the supplement of the angle A.

BOOK IV.

On the Cylinder, the Cone, and the Sphere.

(162)—Def. A *cylinder* is a solid produced by the revolution of a rectangle A B C D, (fig. (Art. 172)) which is conceived to turn on the immovable side A B as an axis.

(163) By this motion the sides B C and A D move in planes which are perpendicular to A B, and their extremities C and D describe circles in those planes with the points B and A as centres, and the lines B C and A D as radii. The side D C evidently describes a surface concave towards the fixed central line A B.

(164) What has been just stated of the sides B C and A D is equally true of any other line perpendicular to A B, which also moves in a plane perpendicular to A B, and its extremity describes a circle on that plane, and which circle is also on the surface of the cylinder. Hence it follows, that every section of the cylinder perpendicular to the line A B is a circle equal to the circular ends A and B.

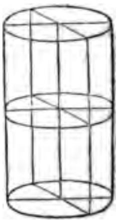
(165) Def.—The line A B is called the *axis* of the cylinder, and the circular ends are called its *bases.*

(166) Cor. 1.—Every section of a cylinder by a plane through the axis is a rectangle equal to twice the generating rectangle A B C D.

(167) Cor. 2.—If from any point C in the circumference of the base of a cylinder a straight line be drawn perpendicular to the plane of the base, that line must be in the cylindrical surface. For it coincides with the side of the generating rectangle when the extremity of that line is at the point C.

(168) Cor. 3.—The right line C D is the intersection of the cylindrical surface with every plane through C parallel to the axis A B or perpendicular to the base. Hence it appears, that the intersection of a cylindrical surface, and a plane which is parallel to its axis, is a right line parallel to the axis; or, as the plane will meet the cylindrical surface twice, it intersects it in two right lines parallel to the axis, and therefore to each other.

The intersections of this plane with the bases of the cylinder are parallel to each other and perpendicular to the axis, and therefore also perpendicular to the intersections with the cylindrical surface. Hence the entire intersection of the plane with the cylinder is a rectangle, whose sides are parallel and perpendicular to the axis.



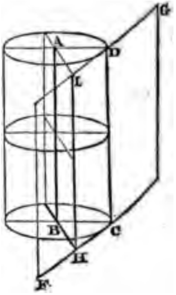
(169) Cor. 4.—It is evident from (167) that a right line drawn through any point in the cylindrical surface parallel to its axis is wholly in the surface.

(170) Def.—Such a line is called a *side* of the cylinder.

Proposition I.

(171) If a plane A B C D be drawn through the axis of a cylinder, and another at right angles F C D G to this and passing through the side C D, the plane F C D G will be entirely outside the cylindrical surface except in the line C D in which it meets it.

For let any other plane A I H B be drawn through the axis of the cylinder intersecting the plane F C D G. In the right angled triangle A D I the hypotenuse A I is greater than the side A D. Since A I is greater than the radius of the circular base, the point I must be outside the cylinder, and the same may be proved of H, and every point in the line H I, and, in general, for every line in the plane F C D G parallel to C D. Hence every part of the plane except the line C D lies outside the cylinder.

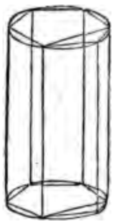


(172) Def.—Such a plane is called a *tangent plane* to the *cylindrical surface.*

(173) Cor. 1.—Hence all tangent planes are parallel to the axis, and their lines of contact are sides of the cylinder.

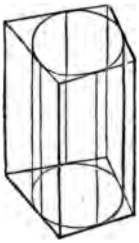
(174) Cor. 2.—Tangent planes which pass through the extremities of the same diameter of the base are parallel, and *vice versâ.*

(175) If the base of a cylinder be divided at three or more points, and sides be drawn through these points, and produced to the opposite base, these sides will divide the opposite base similarly with the first; and if planes be drawn through every



pair of adjacent sides, these planes will form a prism whose sides are those of the cylinder passing through the points of division, and whose bases are formed by the chords of the arcs into which the circular bases of the cylinder are divided. Such a prism is said to be *inscribed* in the cylinder.

(176) If several planes touch the same cylinder intersecting each other, and also the planes of the bases of the cylinder produced, their intersections with the planes of the bases will be tangents to the bases themselves, and the planes may be so disposed, that these tangents shall form polygons circumscribing the bases. Hence a prism will be formed of which the tangent planes are lateral faces, and the polygons are bases.



Such a prism is said to *circumscribe* the cylinder.

(177) It is evident, that both the *volume* and *surface* of a cylinder are greater than those of any *inscribed* prism, and are *less* than those of any *circumscribed* prism.

(178) This observation is applicable to the surfaces, whether the bases be considered as parts of them or not.

(179) It is also evident, that as the number of sides of the inscribed or circumscribed prism is increased, the difference between its volume or surface and that of the cylinder is diminished, and that the sides may be so increased in number as to render this difference less than any given magnitude.

PROPOSITION II.

(180) If a cylinder and right prism have equal bases and altitudes, they will have equal volumes.

Let V be the volume of the cylinder, and V′ that of the prism. If they be not equal, V must either be greater or less than V′.

First. Let V be greater than V′. Let a prism be inscribed in the cylinder V, such that the difference between its volume and that of the cylinder shall be less than the difference between V and V′, and let the volume of this prism be P. It follows that P is greater than V′; but since these prisms have equal altitudes, the base of P must be greater than the base of V′, and therefore greater than the base of V. But the base of P is a polygon inscribed in the base of V, and therefore cannot be greater than the base of V; hence the volume V cannot be greater than V′.

Secondly. Let V be less than V′. Let a prism be circumscribed round the cylinder V, such that the difference between it and the cylinder shall be less than the difference between V and V′, and let the volume of this prism be P. Then V′ is greater than P, and therefore the base of V′ is greater than the base of P, and hence the base of the cylinder is greater than that of its circumscribed prism, which cannot be. Hence V cannot be less than V′, and since it can neither be greater than V′, or less, they must be equal.

(181) COR. 1.—Hence the volume of a cylinder is expressed numerically by the product of its base and altitude.

(182) COR. 2.—Let a be the altitude of a cylinder, and r the radius of its base, and let π be the number which expresses the approximate ratio of the circumference of a circle to its diameter. Then the area of the base is πr^2, and the volume of the cylinder is $\pi r^2 a$.

(183) COR. 3.—Since the areas of circles are as the squares of their radii, or diameters, it follows that the volumes of cylinders are as the products of their altitudes and the squares of their diameters, or in a ratio compounded of their altitudes and the squares of their diameters.

(184) COR. 4.—The volumes of cylinders with equal bases are as their altitudes, and those with equal altitudes are as their bases.

(185) DEF.—*Similar cylinders* are those whose altitudes are proportional to their diameters.

(186) COR. 1.—The volumes of similar cylinders are as the cubes of their altitudes or diameters.

(187) COR. 2.—In equal cylinders the bases and altitudes are reciprocally proportional, and *vice versâ.*

(188) COR. 3.—In equal cylinders the altitudes are inversely as the squares of the diameters, and *vice versâ.*

PROPOSITION III.

(189) If a cylinder and right prism have equal altitudes and isoperimetrical bases, they will have equal convex surfaces.

(By the *convex surfaces* is meant those parts of the surfaces which are included between the bases.)

Let S be the surface of the cylinder, and S′ that of the prism. If these be not equal,

1°. Let S be greater than S′, and let a prism be inscribed in the cylinder, such that the difference between its surface and that of the cylinder shall be less than the difference between S and S′, and let the surface of the prism be P. Hence P is greater than S′, and therefore the perimeter of the base of P is greater than that of S′, or than that of the given cylinder; that is, the perimeter of a polygon inscribed in a circle is greater than the circle itself, which is absurd.

2°. Let S be less than S′; by reasoning precisely similar to that used in Prop. II. we may prove, that if this were the case, the perimeter of a polygon circumscribed round a circle is less than that of the circle itself.

Hence it follows that S and S′ are equal.

Proposition IV.

(190) The cylindrical surface of a cylinder is equal to the rectangle under its altitude and the circumference of its base.

For it is equal to a prism with an isoperimetrical base, (189).

(191) Cor. 1.—A cylindrical surface is represented numerically by the product of its altitude into the circumference of its base.

(192) Cor. 2.—Let a and r represent the altitude and radius, as in (182). Then $S = 2 \pi r a$.

(193) Cor. 3.—Since the circumferences of circles are as their diameters, cylindrical surfaces are as the rectangles under their altitudes, and the diameters of their bases.

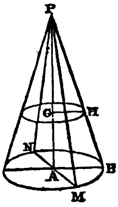
(194) Cor. 4.—Cylindrical surfaces with equal bases are as their altitudes, and with equal altitudes are as the circumferences of their bases.

(195) Cor. 5.—The surfaces of similar cylinders are as the squares of their diameters or altitudes.

(196) Cor. 6.—In equal cylindrical surfaces the diameters and altitudes are reciprocally proportional, and *vice versâ*.

(197) Def.—A *cone* is a solid produced by the revolution of the hypotenuse of a right angled triangle about one of the sides as a fixed axis.

Let P B be the hypotenuse, and P A the fixed axis. The side A B moving in a plane at right angles to P A describes in that plane a circle B M N. This circle is called the *base* of the cone. A convex surface, called the *conical surface*, is described by the motion of the hypotenuse; and as this line, in all its successive positions, must be entirely in the conical surface, it follows that every right line drawn from P to a point in the circumference of the base must be entirely in the conical surface.



The point P is called the *vertex*, and every right line drawn from that point to the circumference of the base is called a *side* of the cone.

(198) Def.—The line P A is called the *axis* of the cone.

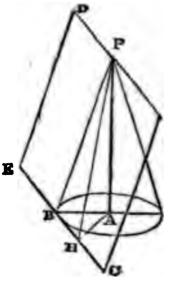
(199) Cor. 1.—It is evident, therefore, that if any plane pass through the vertex and intersect the conical surfaces, the intersections will be two sides of the cone; and as the intersection with the base will be a chord of the circle, the whole intersection will be a triangle M P N.

(200) Cor. 2.—What has been observed of the base A B of the generating triangle, is also applicable to any perpendicular G H to the axis terminated in the hypotenuse of the generating triangle. Such a line G H moves in a plane perpendicular to P A, and its extremity H describes a circle. Hence it follows, that every section of a cone by a plane parallel to the base is a circle, the centre of which is in the axis, and the radius G H of which is to that A B of the base, as their distances from the vertex, that is, as P G : P A.

Proposition V.

(201) **If a plane P A B be drawn through the axes of a cone intersecting the conical surface in the side P B, and through P B another plane C B P D be drawn at right angles to the former, this plane will lie entirely outside the conical surface, except in the line P B, in which it meets it.**

For let any other plane P A H be drawn through the axis P A and intersecting the plane C B P D. In the right angled triangle A B H the hypotenuse A H is greater than the side A B or the radius of the base, and therefore the point H lies outside the base. The same may be proved of every section parallel to the base, and therefore it follows that the plane C B P D meets the conical surface only in P B, lying elsewhere wholly outside it.



(202) Def.—Such a line is called a *tangent plane* to the conical surface, which it touches in the line P B.

(203) Cor. 1.—Hence all tangent planes pass through the vertex, and the lines of contact are *sides* of the cone.

(204) If the circumference of the base be divided at three or more points, and lines be drawn from these to the vertex P, as also lines joining the points of division so as to form a polygon inscribed in the base, these lines will form the edges of a pyramid whose base is the polygon inscribed in the base, and whose sides are the lines drawn to the vertex, and all of which are sides of the cone. Such a pyramid is said to be *inscribed* in the cone.

(205) If several tangent planes be drawn to the same cone intersecting each other, and also the plane of the base of the cone produced, their intersections with the plane of the base will be tangents to the base itself, and the planes may be so disposed that these tangents shall form a polygon circumscribing the base. This polygon will be the base of a pyramid whose lateral faces are the tangent planes. Such a pyramid is said to *circumscribe* the cone.

(206) It is evident that both the volume and surface of the cone are greater than those of any inscribed, and less than those of any circumscribed pyramid.

This observation is applicable to the surfaces whether the bases be parts of them or not.

(207) It is also evident, that the number of sides of the bases of the inscribed or circumscribed pyramid may be increased until the difference between its volume or surface and that of the cone shall be less than any given magnitude.

PROPOSITION VI.

(208) If a cone and pyramid have equal bases and equal altitudes they will have equal volumes.

This proposition is proved in exactly the same manner as (180). In fact, the same words may be used here, changing *cylinder* into *cone*, and *prism* into *pyramid*.

(209) Cor. 1.—Hence the volume of a cone is expressed numerically by one-third of the product of the base and altitude.

(210) Cor. 2.—A cone is one-third of a cylinder on the same base and in the same altitude.

(211) Cor. 3.—If a be the altitude, and r the radius of the base, $\frac{1}{3} \pi r^2 a$ is the volume (182).

(212) Cor. 4.—The volumes of cones are as the rectangle under their altitudes, and the diameters of their bases.

(213) Def.—*Similar cones* are those whose axes are as the radii of their bases.

(214) Cor. 5.—The volumes of similar cones are as the cubes of their altitudes or diameters.

(215) Cor. 6.—Cones with equal bases are as their altitudes, and with equal altitudes are as their bases.

(216) Cor. 7.—If the volumes of cones be equal, their bases and altitudes are reciprocally proportional, and *vice versâ*. Also if the volumes be equal, their altitudes and the squares of the diameters of their bases are reciprocally proportional, and *vice versâ*.

Proposition VII.

(217) The surface of a circumscribed pyramid, exclusive of its base, is equal to half of the rectangle under the side of the cone and the perimeter of the base.

For let P B (fig. Art. (201)) be the line of contact of one of the triangular faces, and let E C be the corresponding side of the base, and P A the axis of the cone. The plane P A B is perpendicular to the plane of the base, and also to the plane D B C. Hence the intersection E C of these planes is perpendicular to the plane P A B, and therefore perpendicular to P B. Hence the area of the triangle E P C is equal to half of the rectangle under the side P B of the cone, and the side E C of the polygonal base of the pyramid. The same being true for every triangular face of the pyramid, it follows that the sum of its triangular faces is equal to half the rectangle under the side of the cone, and the perimeter of the base of the pyramid.

(218) Cor.—Hence the surfaces of circumscribed pyramids are as the perimeters of their bases.

Proposition VIII.

(219) A conical surface is equal to half of the rectangle under the side of the cone and the circumference of its base.

If the conical surface be not equal to half this rectangle, let any other conical surface having the same vertex and axis and its base on the same plane be equal to it. The base of this other cone being concentrical with that of the given one, must be either contained within the base of the given cone, or must contain the base of the given cone within it.

First. Suppose that it is contained within the base of the given cone. Let the surface of the given cone be S, and let the side of the given cone be *s*, and the circumference of its base *c*.

The surface of the lesser cone will then be $\frac{1}{2} s \times c$. Let a polygon be circumscribed round the base of the lesser cone, so as to be contained within the base of the greater. If this polygon be the base of a pyramid circumscribing the lesser cone, its surface will be $\frac{1}{2} s' \times c'$, c' being the circumference of its base, and s' the side of the lesser cone; and this surface will be greater than that of the lesser cone, and less than that of the greater; that is, $\frac{1}{2} s' \times c'$ is greater than $\frac{1}{2} s \times c$, and less than S. But s' the side of the lesser cone is less than s, the side of the greater, and c' the perimeter of the included polygon is less than c the circumference of the circle which includes it. Therefore $\frac{1}{2} s' \times c'$ is less than $\frac{1}{2} s \times c$; but it was already proved to be greater than it, which is absurd. Therefore the base of the cone whose surface is equal to $\frac{1}{2} s \times c$ is not contained within that of the given cone.

Secondly. Let it contain the base of the given cone within it. Let a polygon be circumscribed round the base of the given cone, so as to be included within the greater base. As before, let this polygon be the base of a pyramid circumscribing the given cone, and let the circumference of the base of this pyramid be c', its surface will then be $\frac{1}{2} s \times c'$, and that of the greater cone which includes it, and is therefore greater than it, is $\frac{1}{2} s \times c$. But c' the perimeter of the polygon is greater than c the circumference of the circle which it circumscribes; and therefore $\frac{1}{2} s \times c'$ is greater than $\frac{1}{2} s \times c$, the contrary of which has just been proved. Hence the base of the cone whose surface is equal to $\frac{1}{2} s \times c$ can neither be within nor without that of the given cone, and therefore must coincide with it. The surface of the cone is therefore equal to half the rectangle under its side, and the circumference of its base.

(220) Cor. 1.—The surface of a cone is represented numerically by half the product of its side, and the circumference of its base.

(221) Cor. 2.—The surface of a cone is half that of a cylinder on the same base and with an equal side.

(222) Cor. 3.—If a be the side, and r the radius of the base, the surface $= \pi r a$.

(223) Cor. 4.—Conical surfaces are as the rectangles under their sides and the diameters of their bases.

(224) Cor. 5.—Conical surfaces with equal sides are as the diameters of their bases, and those with equal diameters are as their sides.

(225) Cor. 6.—Similar conical surfaces are as the squares of their sides, or the diameters of their bases.

(226) Cor. 7.—Equal conical surfaces reciprocate their sides and diameters.

(227) Cor. 8.—The surface of a cone is equal to a triangle whose altitude is the side and whose base is the circumference of the base; or it is equal to the area of a sector whose radius is the side, and whose arc is the circumference of the base.

Proposition IX.

(228) **The volume of a truncated cone is equal to the sum of the volumes of three cones, having the same altitude as the truncated cone, and having bases two of which are equal to those of the truncated cone, and the third is a mean proportional between them. Also the conical surface is equal to the rectangle under the side, and half the sum of the circumferences of the bases.**

The first part is evident from (143) and (208).

Suppose the truncated cone completed. Let the entire side be S, and let that which belongs to the upper part be S′, and let the corresponding circumferences be C and C′, the surface will then be $\frac{1}{2}(S \times C - S' \times C')$. But $S : S' :: C : C'$, and $S - S' = x$, x being the side of the truncated cone. By the proportion we easily obtain

$$S = \frac{C}{C - C'}(S - S') = \frac{C x}{C - C'},$$

and hence we find

$$S' = \frac{C' x}{C - C'}.$$

Substituting these values of S and S′ in the expression for the surface it becomes

$$\frac{1}{2}\left(\frac{C^2 - C'^2}{C - C'}\right)x = \frac{1}{2}(C + C')\,x,$$

that is, the rectangle under the side x, and half the sum of the circumference of the bases.

(229) Schol.—The several results to which we have arrived, respecting cylindrical and conical surfaces, may be obtained by a more expeditious and obvious process; although, perhaps, not so rigorous an one as might be desired. It is evident, if a rectangle, whose base is equal to the circumference of the base of a cylinder, and whose altitude is equal to the side of the cylinder,

be described upon a thin flexible plane surface such as paper; and the side of this rectangle, which is equal to that of the cylinder, be applied to it, the rectangle may be rolled round the cylinder so as to cover it exactly. Thus it appears, that a cylindrical surface is nothing more than a plane rectangle which is bent into a round form in one direction. This illustrates the result of (190).

Again, if a circular sector be similarly described on paper, so that its radius shall be equal to the side of a cone, and its arc equal to the circumference of the base; this sector may be rolled upon the conical surface, so as to cover it exactly in the same way as with the cylinder (227).

Cylindrical and conical surfaces belong to an extensive class of surfaces which are distinguished by the general property to which we have just alluded, *viz.* that a plane flexible surface may be applied to them, so as exactly to fit them, or to be in every point in contact with them. Such surfaces are called by the general name of *developable surfaces*, and form an important part in the higher departments of geometry.

The student may, perhaps, conceive that a flexible plane can be applied to any surface. He will, however, see his mistake, if he attempts applying a piece of paper to a globular surface. It will be found to gather into folds, and some parts of it to overlay others.

In elementary geometry it is not usual to extend our investigations beyond *right cylinders* and *right cones*. There are, however, *oblique* cylinders and cones whose axes are not perpendicular to their bases, and which enjoy properties very nearly the same as those which we have established in the present book.

(230) DEF.—A *sphere* is a solid terminated by a curved surface, all the points of which are equally distant from a certain point within it, called its *centre.*

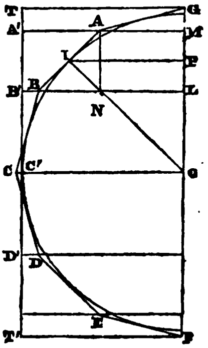
A sphere may be conceived to be produced by the revolution of a semicircle round its diameter; for the surface described by the semicircle in this motion will evidently have all its points equally distant from the centre of the generating semicircle.

We shall confine our investigations here to the magnitude of the *surface* and *volume* of the sphere. For the geometrical properties of the sphere in general, the student is referred to the Treatise on Spherical Geometry contained in the first three sections of the second part of my TRIGONOMETRY.

LEMMA.

(231) Let A B C D, &c. be a regular polygon with an even number of sides circumscribed round a

circle whose centre is O, and let G F be a diagonal of the polygon dividing it into two equal parts and passing through the centre O of the circle. If the figure be supposed to revolve round G F, each side as A B will describe the surface of a truncated cone, which will be equal to a cylindrical surface whose altitude is A N or M L, and whose radius is that of the circle.



For the conical surface described by A B is equal to half the sum of the circumferences described by the points A and B multiplied by the line A B (228). Draw O I to the point of contact I of A B. The point I being the middle point of A B, the circumferences described by the points A, I, and B are in arithmetical progression; and therefore the circumference described by the point I is half the sum of those described by the points A and B. Hence the conical surface described by A B is equal to the line A B multiplied by the circumference described by the point I. Let the circumference described by the point I be c, and let the circumference of the circle whose centre is O and radius O I be C, and we have

$$c : \mathrm{C} :: \mathrm{IP} : \mathrm{IO}.$$

But by the similar right angled triangles B A N and O I P we have

$$\mathrm{IP} : \mathrm{IO} :: \mathrm{AN} : \mathrm{BA},$$

$$\because c : \mathrm{C} :: \mathrm{AN} : \mathrm{BA},$$

$$\because c \times \mathrm{BA} = \mathrm{C} \times \mathrm{AN} = \mathrm{C} \times \mathrm{ML},$$

that is, the circumference described by I multiplied by the line B A, is equal to the circumference of the revolving circle multiplied by M L; but the former is equal to the conical surface described by the line A B, and the latter is equal to a cylindrical surface whose base is equal to the revolving circle, and whose altitude is M L.

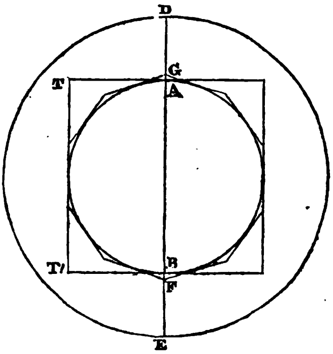
(232) Cor.—Let a tangent T T' to the circle parallel to G F be drawn, and from G and F draw G T, F T' perpendicular to G F, and meeting this tangent. By the revolution of the figure round G F the tangent T T' will describe a cylindrical surface having G F for its axis. Let M A, L B, &c. be produced to meet

T T′ at A′, B′, &c. The cylindrical surface described by A′ B′ will be equal to its altitude A′ B′ or M L multiplied by the circumference described by B′, which is a circle whose radius is B′ L which is equal to O I. Hence the cylindrical surface described by A′ B′ is equal to the conical surface described by A B. In the same manner the cylindrical surface described by B′ C′ is equal to the conical surface described by B C, and so on. It follows, therefore, that the sum of the conical surfaces described by any number of sides of the polygon A B, B C, C D, &c. is equal to the part A′ D′ of the cylindrical surface described by T T′ corresponding to those sides, and that the surface described by the entire contour G A B C D E F of the polygon is equal to the entire cylindrical surface T T′.

Proposition X.

(233) The surface of a sphere is equal to that of the circumscribed cylinder.

The surface of the cylinder T T′ is not greater than that of the sphere, for if it were, let it be equal to the surface of the sphere whose diameter is D E greater than A B. Let a regular polygon with an even number of sides be circumscribed round the circle A B, and so as to be contained within the circle D E. The surface produced by the revolution of this polygon on F G is equal to a cylindrical surface whose altitude is F G, and base is a circle equal to A B. Hence this surface is greater than that of the cylinder T T′, since its base is the same and it has a greater altitude, and therefore it is greater than the surface of the sphere D E; but it is contained within the surface of this sphere, and is therefore less than it, which is absurd. Hence the surface of the cylinder T T′ is not greater than that of the sphere.



The circumscribed cylindrical surface is not less than that of the sphere; for if it were, let the cylindrical surface circumscribing the sphere D E be equal to the surface of a lesser sphere A B. Let a polygon with an even number of sides be circumscribed round the circle A B, and so as to be contained within the circle D E. It then follows, as before, that the surface of this polygon is equal to that of a cylinder which is less than the cylinder circumscribed round D E, but it is greater than the surface of the sphere A B which it circumscribes, and which is equal to the cylindrical surface circumscribed round D E. Hence it follows, that the cylindrical surface circumscribed round a sphere is not less than the surface of the sphere.

Since the cylindrical surface is neither greater nor less than the surface of the sphere, it must be equal to it.

(234) Cor. 1.—The cylindrical surface is equal to the circumference of its base, which is a great circle of the sphere, multiplied by its altitude, which is a diameter of the sphere. Hence the surface of the sphere is equal to the circumference of one of its great circles multiplied by its diameter. But the circumference of the great circle, multiplied by its radius, is equal to twice the area of the circle, and therefore, when multiplied by the diameter, is four times the area. Hence 'the surface of a sphere is equal to four times the area of one of its great circles.'

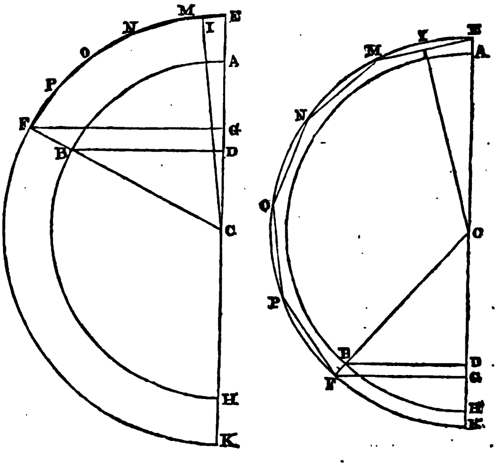
(235) Cor. 2.—Hence the surfaces of spheres are as the squares of their radii or diameters.

Proposition XI.

(236) If a cylinder be circumscribed about a sphere, and they be intersected by any two parallel planes, perpendicular to the axis of the cylinder, the parts of the cylindrical surfaces intercepted between the planes will be equal.

Let E F be any arc of the circle described with the radius C E, and let F G be drawn perpendicular to the radius C E. The surface of the spherical segment described by the revolution of the arc E F round C E, is equal to the surface of a cylinder whose altitude is E G, and whose base is a circle with the radius C E.

For, first, let this spherical surface be less than that of the cylinder, and let it be



equal to the surface of a cylinder having the altitude E G, and a base with the lesser radius C A. In the arc E F inscribe a portion of a regular polygon E M N O P F, whose sides shall not reach the arc A B, and draw C I perpendicular to one of the sides of the polygon. By (231) the surface generated by the revolution of the polygon will be equal to that of a cylinder whose altitude is E G, and the radius of whose base is C I. Hence the surface produced by this polygon must be greater than the surface of a cylinder whose altitude is E G, and whose radius is AC. But this last is, by hypothesis, equal to the surface produced by the revolution of the arc E F. From whence it follows, that the spherical surface produced by the revolution of the arc E F is less than the surface produced by the polygon inscribed in it; but the former surface includes the latter entirely within it, and therefore cannot be less than it. Hence it follows, that the surface of the spherical segment E F is not less than the surface of the cylinder whose altitude is E G, and whose radius is E C.

Next let the spherical surface be greater than that of the cylinder. Let the proposed spherical surface be that which would be produced by the revolution of the arc A B round A C. We are to prove that this surface is not greater than that of a cylinder whose altitude is A D, and whose radius is A C. If possible, let it be greater than this cylindrical surface. But if the surface generated by the arc A B be greater than that of the cylinder whose altitude is A D and radius A C, for the same reason the surface produced by the arc B H must be greater than that of the cylinder whose altitude is D H and whose radius is A C; and hence the whole surface of the sphere would be greater than that of the circumscribing cylinder, which is contrary to what was proved in (233). Hence the surface generated by the arc A B is not greater than a cylindrical surface whose altitude is A D and whose radius is A C.

Hence it appears, that any plane drawn intersecting a sphere and its circumscribing cylinder parallel to the bases of the cylinder, divides the cylindrical and spherical surfaces into parts which are equal each to each. Hence, if two such planes be drawn, the cylindrical and spherical surfaces which they include between them will be the differences between the equal cylindrical and spherical surfaces which they cut off towards either base of the cylinder, and therefore those differences are equal.

(237) Cor. 1.—The spherical surface included between two parallel planes intersecting the sphere is therefore equal to the circumference of a great circle, multiplied by the perpendicular distance between the planes.

(238) Cor. 2.—Let the distance between the planes be a, and let r be the radius of the sphere. The circumference of a great circle

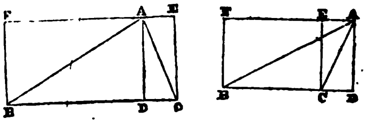
is $2\pi r$, and the surface between the planes is $2\pi r a$. If r' be a mean proportional between $2r$ and a, we have $2\pi r a = \pi r'^2$. Hence the surface is equal to the area of a circle whose radius is a mean proportional between the distance between the planes and the diameter of the sphere.

(239) Cor. 3.—If one of the planes be a tangent plane, the surface becomes that of a spherical segment, and the mean proportional between the diameter and perpendicular is the chord E F of the generating arc. Therefore the surface of a spherical segment is equal to a circle whose radius is the chord of half the arc formed by a section of the segment by a plane through its axis.

Proposition XII.

(240) If the triangle B A C, and the rectangle B C E F, having the same base and the same altitude, revolve together round their common base B C, the solid described by the revolution of the triangle will be one third of the cylinder described by the revolution of the rectangle.

Draw the perpendicular A D. The cone described by B A D is one third of the cylinder described by B E A D, and in like manner the cone described by C A D is one third of the cylinder described by C E A D (210). When the perpendicular A D falls *within* the base, the solids in question are the sums of these cones and cylinders, and when it falls *without* the base A D they are their differences. In either case the truth of the proposition is therefore apparent. If the perpendicular fall on the extremity of the base, the solids in question will be simply a cylinder and cone having the same base and altitude, and the proposition is reduced to (210).

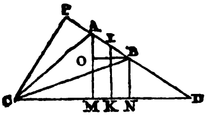


(241) Cor.—Since the cylinder is equal to $\pi \cdot AD^2 \times BC$, the solid described by the triangle will be $\frac{1}{3}\pi \cdot AD^2 \times BC$.

Proposition XIII.

(242) The triangle C A B being supposed to revolve round any line C D passing through the vertex C, to determine the volume of the solid produced by its revolution.

Produce the side A B until it meet the axis in D, and draw the right lines A M and B N perpendicular to the axis C D.



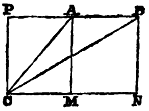
The volume of the solid described by the triangle C A D is $\frac{1}{3}\pi . AM^2 \times CD$ (240). The solid described by the triangle C B D is in like manner $\frac{1}{3}\pi \times BN^2 \times CD$. Hence the difference of these solids, or the solid described by A B C is $\frac{1}{3}\pi . (AM^2 - BN^2) \times CD$.

Let I be the middle point of A B and draw I K perpendicular to C D. Hence A M, I K, and B N are in arithmetical progression, and therefore $2 IK = AM + BN$. Also, if B O be drawn parallel to C D, $AO = AM - BN$. Therefore $2 IK \times AO = (AM + BN) \times (AM - BN) = AM^2 - BN^2$. Hence the solid described by the triangle C A B is equal to $\frac{2}{3}\pi \times IK \times AO \times CD$. If C P be drawn perpendicular to A B, the triangles D C P and A B O are similar, and we have therefore

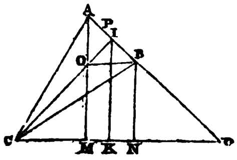
$$AO : AB = CP : CD,$$
$$\therefore AO \times CD = AB \times CP.$$

But $AB \times CP$ is twice the area C A B. Hence the solid described by C A B is equal to $\frac{2}{3}\pi \times IK \times$ area C A B; or, which is the same, it is equal to $\frac{2}{3}$ C A B multiplied by the circumference whose radius is I K. Hence "the volume of the solid described by the revolution of the triangle C A B is equal to two thirds of the area of the triangle C A B multiplied by the circumference traced by I, the middle point of the base."

We have, in the preceding proof, supposed that A B produced will meet the axis C D. The same result will, however, be obtained if A B be parallel to the axis. In this case the volume of the cylinder described by A M N B is equal to $\pi . AM^2 . MN$; the cone described by A C M is equal to $\frac{1}{3}\pi . AM^2 . CM$, and the cone described by B C N is equal to $\frac{1}{3}\pi . AM^2 . CN$. Add the first two volumes, and subtract from their sum the third, and we shall have the volume described by A B C equal to $\pi . AM^2 (MN + \frac{1}{3} CM - \frac{1}{3} CN)$; and since $CN - CM = MN$, the volume is equal to $\pi . AM^2 . \frac{2}{3} MN = \frac{2}{3}\pi . CP^2 . MN$; which is equivalent to the result already found.



(243) Cor. 1.—If the triangle A C B be isosceles the point P will coincide with I, and the area C A B will be equal to $AB \times \frac{1}{2} CI$, and the volume of the solid will be $\frac{2}{3}\pi \times ABC \times IK$, or $\frac{2}{3}\pi \times AB \times IK \times CI$. But the triangles A B O and C I K are similar, and therefore



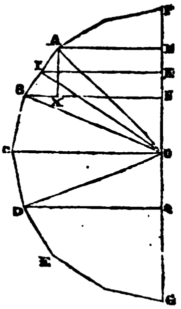
$$AB : BO = CI : IK,$$
$$\text{or } AB : MN = CI : IK,$$
$$\therefore AB \times IK = MN \times CI.$$

Hence the solid described by the isosceles triangle **A B C** is equal to $\frac{2}{3}\pi \times MN \times CI^2$.

Proposition XIV.

(244) Let A B, B C, C D, ... be several sides of a regular polygon, O its centre, and O I the radius of the inscribed circle; if the polygonal sector A O D, lying all on the same side of the diameter F G, be supposed to revolve round this diameter, the volume of the solid described by it will be equal to $\frac{2}{3}\pi . OI^2 . MQ$; M Q being that part of the axis included between perpendiculars A M, D Q from the extremes of the polygonal sector.

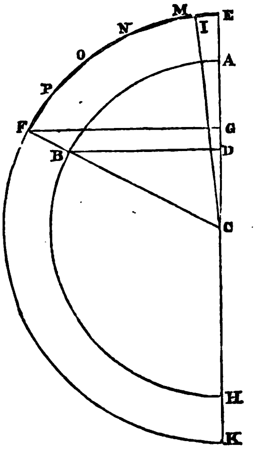
Since the polyon is regular, the several triangles A O B, B O C, ... are equal and isosceles. By (243) the volume described by the triangle A O B is equal to $\frac{2}{3}\pi . OI^2 . MN$. In the same manner the volume described by B O C is equal to $\frac{2}{3}\pi . OI^2 . NO$, and so on. Hence the whole solid described by the polygonal sector being the sum of these is equal to $\frac{2}{3}\pi . OI^2 . (MN + NO + \ldots.)$, or $\frac{2}{3}\pi . OI^2 . MQ$.

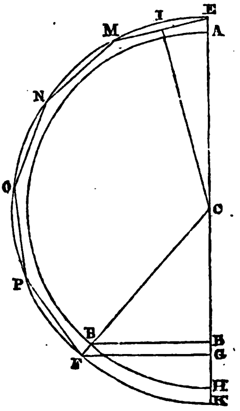


Proposition XV.

(245) The volume of a spherical sector is equal to its spherical surface multiplied by a third of the radius, and the volume of the whole sphere is equal to the surface of the sphere multiplied by a third of the radius.

Let A B C be the circular sector which by its revolution round A C describes the spherical sector. The surface of the spherical segment described by the arc A B is equal to 2π . A C . A D (236). Now the volume of the spherical sector A C B is equal to $\frac{2}{3}\pi$. A C^2 . A D, that is, the surface of the spherical segment multiplied by a third of the radius.





First, suppose that $\frac{2}{3}\pi$. A C^2 . A D is greater than the sector A C B and equal to the sector E C F. In the arc E F inscribe a portion of a regular polygon, so that its sides shall not meet the arc A B. Let the polygonal sector E C F be supposed to revolve at the same time with the arc E F round E C. Let C I be the radius of the circle inscribed in the polygon, and let F G be drawn perpendicular to E C. The volume described by the polygonal sector will be equal to $\frac{2}{3}\pi$. C I^2 . E G . (244); but C I is greater than A C, and E G is greater than A D; for if A B and E F be drawn, the similar triangles E F G and A B D give

$$EG : AD = FG : BD = CF : CB,$$

and therefore E G is greater than A D.

Hence it follows, that $\frac{2}{3}\pi$. C I^2 . E G is greater than $\frac{2}{3}\pi$. C A^2 . A D. The former is equal to the volume described by the polygonal sector, and the latter is, by hypothesis, that of the spherical sector E C F. Hence the volume described by the polygonal sector is greater than that of the spherical sector E C F, of which it is a part, which is absurd. Hence the spherical surface multiplied by a third of the radius is not greater than the volume of the spherical sector.

Secondly, this product is not less than the volume of the spherical sector. Let E C F be the circular sector which by its revolution describes the spherical sector, and suppose, if possible, that $\frac{2}{3}\pi$. C E^2 . E G is equal to a smaller sector A C B. The former construction being retained, the volume of the polygonal sector will be $\frac{2}{3}\pi$. C I^2 . E G. But C I is less than C E, and therefore the volume of the polygonal sector is less than $\frac{2}{3}\pi$. C E^2 . E G, which, by hypothesis, is equal to the volume of the spherical sector

A C B. Hence the volume of the polygonal sector must be less than that of the spherical sector A C B, which is a part of it, which is absurd. Hence the product of the spherical surface by a third of its radius is not less than the volume of the spherical sector.

Hence this product must be equal to the volume of the spherical sector.

If the circular sector A C B be supposed to increase by the increase of the angle A C B, until it becomes equal to a semicircle, the corresponding spherical sector will become equal to the whole sphere. Hence the volume of the entire sphere is equal to the product of its surface by a third of its radius.

(246) Cor. 1.—The volume of a spherical sector is equal to that of a cone whose altitude is the radius, and whose base is equal to the spherical base of the sector; for the volume of the cone is equal to its base multiplied by a third of its altitude.

This may also appear from considering the spherical sector to be formed of an infinite number of cones having the same vertex, equal altitudes, and infinitely small bases, which bases may be conceived to form the spherical base of the sector.

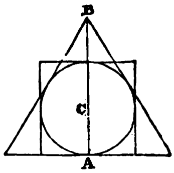
(247) Cor. 2.—The volume of a sphere is equal to that of a cone whose base is equal to the surface of the sphere, and whose altitude is equal to the radius.

(248) Cor. 3.—The volumes of spheres are as the cubes of their radii or diameters. For their surfaces are as the squares of the radii, and these being multiplied by one third of the radii give products which are as the cubes of the radii.

Proposition XVI.

(249) Let a square and an equilateral triangle be circumscribed round the same circle, the base of the equilateral triangle coinciding with a side of the square, and let the whole figure revolve round the altitude B A of the triangle. A sphere will thus be described circumscribed by a cylinder and cone; the entire surfaces of this sphere, cylinder, and cone, are in continued proportion, the common ratio being 2 : 3, and their volumes are also in continued proportion and in the same ratio.

The surface of the sphere is equal to four times one of its great circles (234). The cylindrical surface is equal to this; and as each of the bases of the cylinder is a great circle, the entire surface of the cylinder is equal to six times a great circle. Hence the ratio of the surface of the sphere to the entire surface of the cylinder is $4:6$, or $2:3$.



If the radius C A of the circle be r, half the base of the equilateral triangle will be $\sqrt{3}\,.\,r$; hence the area of the base of the cone will be $3\,r^2\,\pi$. The circumference of the base will be $2\,.\,\sqrt{3}\,.\,r\,\pi$, and the side of the cone is $2\,.\,\sqrt{3}\,.\,r$; therefore the conical surface is $2\,.\,3\,.\,r^2\,\pi$, or $6\,r^2\,\pi$ (222); to which if the base be added, the entire surface of the cone will be $9\,r^2\,\pi$, or nine times the area of a great circle. Thus it appears, that the entire surfaces of the sphere, cylinder, and cone, are respectively equal to 4, 6, and 9 great circles, and are, therefore, in continued proportion, the common ratio being $2:3$.

The volume of the sphere is equal to $\frac{4}{3}\,r^3\,\pi$ (245). The volume of the cylinder is $2\,r^3\,\pi$. The base of the cone being $3\,r^2\,\pi$, and its altitude $3\,r$, its volume is $3\,r^3\,\pi$. Hence the volumes of the three solids are as the numbers $\frac{4}{3}$, 2, 3, or 4, 6, 9, which are in continued proportion, the common ratio being $2:3$.

(250) Cor.—The base of the cone is equal to three great circles, and its conical surface to six. Hence in such a cone the conical surface is double its base.

APPENDIX

No. I.

GEOMETRICAL ANALYSIS.

SECTION I.

Introduction.

(1) ANALYSIS, or *resolution*, is a name given to a species of mathematical investigation, which commencing with the assumption of that which is *sought* as if it were *given*, a chain of relations is pursued which terminates in what is *given* (or may be obtained) as if it were sought. SYNTHESIS, or *composition*, is a process the very reverse of this; being one in which the series of relations exhibited commences with what is given, and ends with what is sought. Consequently *analysis* is the instrument of invention, and *synthesis* that of instruction.

The analysis of the ancients is distinguished from that of the moderns by being conducted without the aid of any calculus, or the use of any principles except those of Geometry, the latter being conducted entirely by the language and principles of Algebra. The ancient is, therefore, called the *Geometrical Analysis.*

The interest which the Geometrical Analysis derives from its antiquity, and from having been the instrument by which the splendid results of the ancient Geometry were obtained, would alone be sufficient to render it an object of attention even after the discovery of the more powerful agency of Algebra. But this is not its only nor its principal claim upon our notice. Its inferiority, compared with the modern analysis, in power and facility, is balanced by its extreme purity and rigour; and though its value as an instrument of discovery be lost, yet it must ever be considered as a most useful exercise for the mind of a student; and it may be fairly questioned, whether it may not be more conducive to the improvement of the mental faculties than the modern analysis, unless the latter be pursued much farther than it usually is in the common course of academical education, in which the student acquires little more than a knowledge of its notation. Newton was fully aware of the advantages attending the cultivation of this branch of mathematical science, and in many parts of his works laments that the study of it has been so much abandoned. He considered, that, however inferior in power and despatch the ancient method might be, it had greatly the advantage in rigour and purity; and he feared, that by the premature and too frequent use of the modern analysis the mind would become debilitated and the taste vitiated. We must however confess, that the pretensions of the ancient method to superior *rigour* do not seem to us to be as well founded as they are sometimes considered. It would be no very difficult matter to expunge the algebraical symbols from a modern investigation, and substitute for them their meaning expressed in the language used in geometrical investigations; but would such a change confer upon them greater rigour, or would it give to the conclusions greater validity? And yet this is precisely what Newton himself has done in many parts of his great work, the *Principia*. His theorems are, evidently, investigated algebraically; but in demonstrating them, the process is disguised by the substitution of lines and geometrical figures for the algebraical species and formulæ. It cannot but excite astonishment, that a man of his extraordinary sagacity could so far deceive himself, as to suppose that by such a proceeding his reasoning acquired greater *rigour*.

But without reference to the modern analysis, we conceive that the ancient method has sufficient claims to our attention on the score of its own intrinsic beauty. It has this further advantage, that we can enter at once upon its most interesting discussions without the repelling task of learning any new language or system of notation.

In the application of the Geometrical Analysis to the solution of problems, or the demonstration of theorems, no general rules nor invariable directions can be given which will apply in all cases. The previous construction to be used, and the preparatory steps to be taken, depend on the particular circumstances of the question, and must be determined by the sagacity of the analyst; and his skill and taste will be evinced in the selection of the properties or affections of the given or sought quantities on which he founds his analysis; for the same question may frequently be investigated in many different ways.

In submitting a *problem* to analysis, its solution, in the first instance, is assumed; and from this assumption a series of consequences are drawn, until at length something is found which may be done by established principles, and which *if done* will necessarily lead to the execution of what is required in the problem. Such is the *analysis*. In the *synthesis*, then, or the *solution*, we retrace our steps; beginning by the execution of the construction indicated by the final result of the analysis, and ending with the performance of what is required in the problem, and which constituted the first step of the analysis.

When a *theorem* is submitted to analysis, the thing to be determined is, whether the statement expressed by it be true or not. In the analysis this statement is, in the first instance, assumed to be true; and a series of consequences is deduced from it until some result is obtained, which either is an established or admitted truth, or contradicts an established or admitted truth. If the final result be an established truth, the theorem proposed may be proved by retracing the steps of the investigation, commencing with that final result, and concluding with the proposed theorem. But if the final result contradict an established truth, the proposed theorem must be false, since it leads to a false conclusion.

These general observations on the nature of the Geometrical Analysis, and the methods of proceeding in it, will be more clearly apprehended after the investigations contained in the subjoined treatise have been examined.

SECTION II.

Problems respecting right lines.

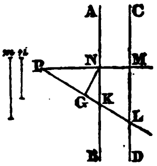
(2) Def.—A point is said to be given when its position is either given or may be determined.

(3) Def.—A *right line* is said to be *given in position* when it is either actually exhibited and drawn, or may be exhibited and drawn by previously established principles.

Proposition.

(4) *To draw from a given point a right line intersecting two right lines given in position, so that the segments between the point and the right lines shall have a given ratio.*

Let the given point be P, A B and C D the right lines given in position, and $m : n$ the given ratio.



Let $PM : PN = m : n$. If any other line as P L be drawn intersecting A B and C D, and a parallel to C D be drawn from N, that parallel will divide P L similarly to P M, and therefore in the required ratio. This parallel may, or may not, coincide with the line N K. First, let us suppose that it does. In that case the two lines given in position will be parallel, and the line P L, or any other line, drawn intersecting them, will be cut similarly to P M, and therefore all such lines will be cut in the required ratio. Hence it appears, that in this case the problem is indeterminate, since every line which can be drawn intersecting the given lines will equally solve it.

Secondly, if the given lines A B, C D be not parallel, let the parallel to C D from N meet P L in G, so that $PL : PG = m : n$. But P L may be drawn, and the point G therefore may be determined; and since the direction of C D is given, the direction of G N is determined, and therefore the point N may be found. Hence, the solution is as follows: let any line P L be drawn. If $PL : PK = m : n$, the problem is solved. If not, let P L be cut at G, so that $PL : PG = m : n$, and from G draw G N parallel to C D, meeting A B in N, and through N draw P N M. Then $PM : PN = PL : PG = m : n$.

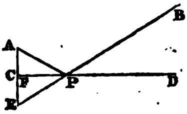
(5) Cor. 1.—The same solution will apply if the line A B be a curve of any kind.

(6) Cor. 2.—If the parallel to C D through G do not meet the line A B, the solution is impossible. If A B be a right line, this happens

when it is parallel to C D. And therefore we conclude in general, that when the two right lines A B and C D are parallel, the problem is either indeterminate or impossible.

PROPOSITION.

(7) ***From two given points to draw to the same point in a right line given in position, two lines equally inclined to it.***



Let the given points be A and B, and let C D be the line given in position. Let P be the sought point, so that the angle A P C shall be equal to the angle B P D.

Produce the line B P beyond P, until P E is equal to P A, and join A E. The angles B P D and E P C are equal; but also (hyp.) B P D and A P C are also equal, therefore the angle A P C is equal to the angle E P C. But also the sides P A and P E are equal, and the side P F is common to the triangles A P F and E P F. Therefore the angles A F P and E F P are equal, and therefore are right angles, and also A F is equal to E F.

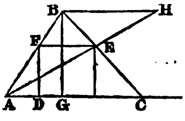
But since A and C D are given, the perpendicular A F is given, and hence the solution of the problem may be derived.

From either of the given points A draw a perpendicular A F to the given right line C D, and produce it through F, until F E is equal to A F, and draw the right line E B meeting the line C D in P. Draw A P, and the lines A P and B P are those which are required. For since A F and F E are equal, and P F common to the triangles A F P and E F P, and the angles A F P and E F P are equal, the angles A P F and E P F are equal. But B P D and E P F are also equal, therefore the angles A P F and B P D are equal.

SCHOLIUM.—If the given points lie at different sides of the given right line, the problem is solved by merely joining the points.

PROPOSITION.

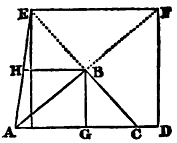
(8) *To inscribe a square in a triangle.*



Let A B C be the triangle, and D F E the required square. Draw the perpendicular B G, and draw A E to meet a parallel B H to A C at H. It is easy to see that D F : F E = G B : B H; for the triangles A F D and A B G, A F E and A B H are respectively similar each to each. Hence, since D F is equal to F E, G B is also equal to B H. But G B is given in magnitude and position, and therefore B H is given in magnitude and position. To solve the problem therefore it is only necessary to draw B H and join A H, and the point E where A H meets B C will be the vertex of the angle of the square.

(9) COR. 1.—It is evident that the same analysis will solve the more general problem, " To inscribe in a triangle a rectangle given in species." **For in this case the ratio B H : B G is given, and therefore B H is as before given in position and magnitude.**

(10) **Schol.**—If B H be drawn equal to B G and on the same side of the vertex with A, then it will be necessary to produce A H and C B, in order to obtain their point of intersection E. In this case, however, D F E will still be a square, for the corresponding triangles will be similar, B G A to F D A, and H B A to E F A. Hence G B : B H = D F : F E.



(11) **Cor. 2.**—In the same manner the more general problem, "To inscribe a rectangle given in species," may be extended.

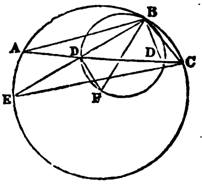
Proposition.

(12) *To draw a line from the vertex of a given triangle to the base, so that it will be a mean proportional between the segments.*

Let A B C be the triangle, and let B D be a mean proportional between A D and D C. Produce B D to E, so that D E shall be equal to B D, and join C E.

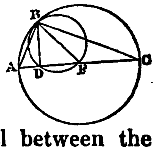
Since A D : B D = E D : D C,

and the angles B D A and E D C are equal, the triangles B D A and C D E are similar. Therefore the angles E and A are equal, and are in the same segment of a circle described on C B. If from the centre of this circle F D be drawn, the angle F D B will be a right angle, and the point D will therefore be in a circle described on F B as diameter. But the point F is given, since it is the centre of a circle circumscribed about the given triangle, and the line F B is therefore given, and the circle on it is as diameter is given, and therefore the point D is given. The solution of the problem is therefore effected by circumscribing a circle about the given triangle, and drawing from its centre to the angle B a radius. On that radius, as diameter, describe a circle; and to a point D, where this circle meets the base, draw the line B D, and it will be a mean proportional between the segments. For the angle B D F in a semicircle is right, therefore B D = D E; and therefore the square of B D is equal to the rectangle under A D and D C.



If the circle on B F intersect A C, there will be two points in the base to which a line may be drawn, which will be a mean proportional between the segments. If this circle touch the base there will be but one such line, and it may happen that the circle may not meet the base at all, in which case the solution is impossible.

If the centre F be upon the base A C, the angle A B C will be right, and the point F itself is one of the points which solve the problem; for in that case A F, B F, and C F are equal. The other point D is the foot of a perpendicular B D from the vertex on the base.

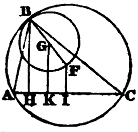


(13) **Cor.**—Hence, in a right angled triangle, the perpendicular on the hypotenuse is a mean proportional between the segments; and it is the only line which can be drawn from the right angle to the hypotenuse which is a mean, except the bisector of the hypotenuse.

Schol.—It has been observed by some elementary writers, that the solution of the problem to draw a line to the base which shall be a mean proportional between the segments is impossible when the vertical angle is acute. That this is erroneous, must be evident from the preceding analysis. For let one circle be described upon the radius of another as diameter. Let any line, as A C, be drawn not passing through F, but intersecting the inner circle; and so that the point of contact B and the centre F shall lie at the same side of it. Draw A B and C B, and also B D. It is evident that B D is a mean proportional between A D and C D, and yet the angle A B C is acute, being in a segment greater than a semicircle.

The possibility of the solution of this problem does not at all depend on the magnitude of the vertical angle. It may be obtuse, right, or acute, and may be equal in fact to any given angle, and yet the solution be possible.

Let it be required to determine the conditions on which the solution is possible. If the circle on B F meet the base, the perpendicular distance of its centre from the base must be less than its radius; that is, less than half the radius of the circle which circumscribes the given triangle. From F and B draw perpendiculars F I and B H on A C, and from the centre of the lesser circle G draw the perpendicular G K. Since G F is equal to G B, G K is equal to half the sum of F I and B H. Hence it follows, that the solution will only be possible when half the sum of F I and B H is not greater than B G, or when the sum of F I and B H is not greater than B F; that is, when the sum of the perpendiculars on the base from the vertex and the centre of the circumscribed circle is not greater than the radius of that circle.



SECTION III.

Propositions respecting circles.

(14) **Problems** of contact of right lines and circles furnished the ancients with an extensive subject for the exercise of the Geometrical Analysis. In general three conditions are necessary to determine a circle. In the class of problems to which we allude, one at least of these conditions is, that it should touch a given right line or a given circle. The other data may be, that it should pass through one or two given points, or that it should have a given radius or centre, or that the locus of its centre should be a given right line or circle. It would not be easy to enumerate all the problems of this class; but by combining the following data for the determination of a circle, a considerable number of them may be found.

To describe a circle

1. Passing through a given point.
2. Passing through two given points.
3. Passing through three given points.
4. Touching a given right line.

5. Touching two given right lines.
6. Touching three given right lines.
7. Touching a given circle.
8. Touching two given circles.
9. Touching three given circles.
10. Having a radius given in magnitude.
11. Having its centre on a given right line.
12. Having its centre on a given circle.
13. Having a given centre.

Every combination of three which can be formed from these data, may be taken as the limiting circumstances in problems for the determination of a circle. In the invention of such problems it should however be observed that 2, 5, 8, and 13 are each to be counted as two data, and 3, 6, 9 are each to be counted as three data. Each of the latter is, therefore, itself sufficient to determine the circle, but each of the former ought to be combined with some one of the data 1, 4, 7, 10, 11, 12.

We cannot here enter at large on this class of problems, we shall therefore confine ourselves to a few examples.

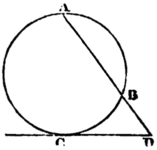
Proposition.

(15) *To describe a circle passing through two given points, and touching a right line given in position.*

If the given points be at different sides of the given line, the solution is manifestly impossible.

Let them then be A, B at the same side of the given right line C D. Let the required circle be A B C, and let A B be produced to meet the right line at D.

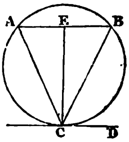
The square of C D is equal to the rectangle A D × D B. But this rectangle is given, therefore the square of C D is given, and therefore C D itself is given in magnitude and position, and hence the point C is given. But also the points A, B being given, the circle through these points A, B, C is given.



The solution, therefore, is effected by producing A B to D, and taking D C equal to a mean proportional between A D and D B, and then describing a circle through A, B, C.

But it may happen, that the line A B is parallel to C D, and will not meet it when produced.

In this case draw A C and B C. The angle B C D is equal to the angle A in the alternate segment, and also equal to the alternate angle B. Hence the angles A and B are equal, and therefore the sides A C and B C are equal. Draw C E perpendicular to A B, and A E and B E are equal. The point E is therefore given, and the perpendicular E C is given in position, and therefore the point C is given.

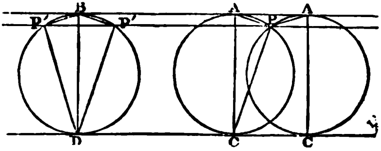


To solve the problem in this case therefore, bisect A B at E, and draw the perpendicular through E, intersecting C D in C. A circle passing through A, B, C will be that which is required.

PROPOSITION.

(16) *To describe a circle passing through a given point, and touching two right lines given in position.*

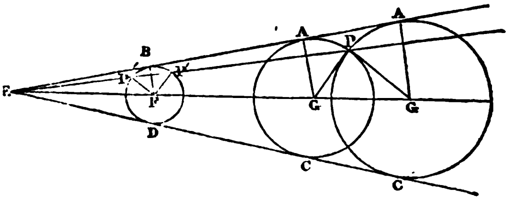
1°. Let the given right lines be parallel. In this case it is necessary that the point should be between them, for otherwise the solution would be impossible.



Let the lines be A B, C D, and the point be P. Let A P C be the required circle, and draw A P and the diameter A C. Through P draw P P′ parallel to the given right lines, and describe any circle B P′ D, touching the right lines at B, D, and intersecting the parallel at P′, and draw P′ B. Since the circle B P′ D may be drawn, the point P′ is given, and therefore the line P′ B is given in magnitude and position. But the triangles B P′ D and A P C are similar, and since B D and A C are parallel, B P′ and A P are parallel. Therefore the line P A is given in direction, and since the point P is given, it is also given in position. Hence the points A and C are given, and therefore the circle A P C is given.

To solve the problem therefore, describe any circle touching the two lines, and draw the parallel through P to meet it at P′. From P′ draw P′ B, and draw P A parallel to it. Draw A C perpendicular to A B, and it will be the diameter of the required circle.

2°. Let the given lines A B, C D intersect at E.



As before, describe any circle B P′ D touching the right lines, and from E draw E P intersecting this circle at P′. Draw the radii G A, G P, F B, and F P′. Since G A is parallel to F B, we have

$$GA : FB = GE : FE.$$

Therefore $$GP : FP' = GE : FE.$$

Therefore $$GP : GE = FP' : FE.$$

Hence the lines G P and F P′ are parallel. But F P′ is given in position, and therefore G P is given in direction, but P is given, and therefore G P is given in position. But the line E G bisects the angle A E C under the given lines, and is therefore given in position, and therefore the point G where it intersects P G is given. Hence the centre G and the radius G P of the required circle are given, and therefore the circle itself is given.

To solve the problem, draw E P, and also E G, bisecting the angle E. Describe any circle B P′ D touching the given right lines, and draw P′ F. Through P draw P G parallel to P′ F, meeting the bisector E G in G. With G as centre and G P as radius, let a circle be described. This circle will touch the right lines. The demonstration is obvious.

It is evident, that in each of the preceding cases there may be two circles drawn, which will solve the problem. This circumstance arises from the line P P′ meeting the circle B P′ in two points. The principle used in the solution of both cases is the same. The parallel in the first case corresponds to the bisector of the angle in the second.

PROPOSITION.

(17) *To describe a circle passing through two given points* (A, B) *and touching a circle given in magnitude and position.*

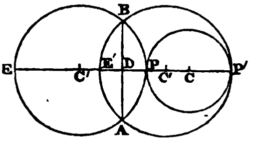
It is evident that if one of the given points be *within*, and the other *without*, the given circle the solution is impossible.

Since the right line joining the centres must pass through the point of contact, it follows that if the right line joining the given points pass through the centre of the given circle, it must be a diameter of the required circle, and consequently in this case the solution is only possible when one of the given points is on the circumference of the given circle. The line joining the given points is then the diameter of the required circle. The only cases that remain to be considered are those in which the given points do not lie in the same right line with the centre of the given circle, and are either both *without* or both *within* the given circle.

Let a right line be drawn from C, the centre of the given circle, to the point of bisection of the line A B joining the given points.

1. Let the points A, B be both *without* the given circle, and the angle C D B be right.

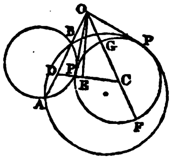
Since A B is bisected at D and P E perpendicular to A B, P E must be a diameter of the required circle, and therefore its middle point C′ is the centre. The rectangle P D × D E is equal to the square of B D, and is therefore given. But P D is given, and therefore D E is also given. Hence, to solve the problem, find a third proportional to P D and D B, and take D E equal to it, and on P E as diameter describe a circle. This will be the required circle. Since the line D P meets the given circle in two points there will be two solutions, in one of which the constructed circle touches the given circle externally, and in the other internally.



2. Let the points A B be both *without* the given circle, and let the angle C D B be acute.

Through the point of contact P draw the common tangent P O, and produce it to meet A B at O. Draw O C, and from O draw O E perpendicular to C D.

The square of P O is equal to the rectangle A O × O B, and also to the rectangle F O × O G. But the former is equal to the difference of the squares of D O and D B, and the latter is equal to the difference of the squares of C O and C G. Therefore these differences are equal, or what is the same, the difference of the squares of O D and O C is equal to the difference of the squares of D B and C G. But these latter quantities are both



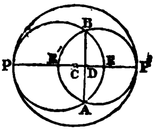
given, the former being half the given line A B, and the latter being the radius of the given circle. Hence the difference of the squares of D O and C O is given, and therefore also the difference of the squares of C E and D E is given. Hence the point of section E may be determined, and the solution of the problem thence effected.

Let the given line C D be divided at E, so that the difference of the squares of C E and D E shall be equal to the difference of the squares of C G and D B; and through E draw E O perpendicular to C D, to meet A B produced at O. From O draw the tangent O P. A circle described through the points A, B and P will solve the problem. The demonstration will be easily obtained by retracing the preceding analysis.

Since two tangents may be drawn from O, there are two circles which will solve the problem, one touching the given circle internally and the other externally.

3. Let the points A B be both *within* the given circle, and let the angle C D B be right.

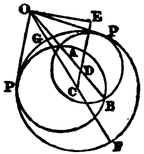
Since C D bisects A B at right angles, it passes through the centre of the required circle. Hence P E (or P′ E′) is a diameter. Therefore D E (or D E′) is a third proportional to P D (or P′ D) and B D; and is therefore given. Hence the problem is solved by two different circles on the diameters P E, P′ E′ in a manner analogous to the first case.



4. Let the points A, B be both *within* the circle, and the angle C D B acute.

Produce the line B A to meet the tangent at O. Draw O C, and perpendicular to C D produced draw O E.

The square of P O is equal to the rectangle F O × O G, and also to B O × O A. The former is equal to the difference of the squares of C O and C G, and the latter to the difference of the squares of D O and D A. Hence the difference of the former squares is equal to the difference of the latter; and therefore the difference of the squares of C O and D O is equal to the difference of the squares of C G and D A. But also the difference of the squares of C O and D O is equal to the difference of the squares of C E and D E. Hence the difference of the squares of C E and D E is equal to the difference of the squares of C G and D A which are given, and therefore the point E is given, from which the solution is derived, as follows.



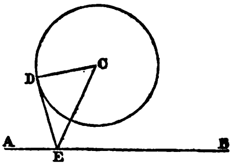
Produce C D to E, so that the difference of the squares of C E and D E is equal to the difference of the squares of C G and D A; and from E draw the perpendicular E O to meet B A produced to O. From O draw the tangents O P, and either of the circles through the points P, A, B will solve the problem. The demonstration will be apparent by retracing the analysis.

In the solution of this problem given in some works, none of the preceding cases, except the second, are included.

PROPOSITION.

(18) *To draw the shortest tangent from a right line* (A B) *given in position, to a circle* (C D) *given in magnitude and position.*

If the right line meet the circle, it is evident that there is no limit to the length of the tangent.



Let D E be a tangent at D. The squares of D E and D C together are equal to the square of C E. Whatever be the position of the points D, E the length of C D is the same; and therefore the square of D E must increase or decrease with that of C E; and therefore if D E be a minimum, C E must also be a minimum. Hence C E is perpendicular to A B when D E is a minimum. Hence the solution is evident.

PROPOSITION.

(19) *Given the three sides* (c, c', c'') *of a triangle, to determine the radius* (R) *of the circumscribed circle.*

Let p be the perpendicular on the side c'' from the opposite angle. By the Elements we have

$$2\,R : c = c' : p.$$

But also

$$c'' \times c' : c'' \times p = c' : p.$$

But $c'' \times p$ is equal to twice the area (A). Hence

$$2\,R : c = c' \times c'' : 2\,A.$$

The sides being known the area is also known, and from this proportion R may be found.

If the lines be given in numbers, by multiplying the means and extremes we have

$$4\,R \times A = c\,c'\,c''$$

$$R = \frac{c\,c'\,c''}{4\,A}.$$

That is, 'The radius of the circumscribed circle is equal to the product of the three sides divided by four times the area.'

SECTION IV.

On Loci.

(20) WHEN a point is required to be determined in a problem with data which are insufficient for its solution, the problem is said to be indeterminate, because the position of the point cannot be found by it. But although the position cannot be absolutely determined, yet it may be so restricted by the conditions which are prescribed in the problem, that it may be known to be on some line, the nature of which may frequently be determined. This line is called the *locus* of the point. This will easily be understood by the following examples: suppose that the base of a triangle were given in magni-

tude and position, and that its area were given in magnitude, to determine its vertex. In this case, it is evident that the problem is indeterminate, since innumerable triangles may be constructed on each side of the given base having equal areas. But since the area is equal to the rectangle under the perpendicular and half the base, it follows that the perpendiculars from the vertices of all these triangles on the base must be equal, and therefore these vertices must all lie on parallels to the base at such a perpendicular distance that the rectangle under it and half the base shall be equal to the given magnitude.

The *locus* of the vertex is therefore in this case two right lines parallel to the base, and at equal perpendicular distances at opposite sides of it.

If the base of a triangle be given in magnitude and position, and the vertical angle be given in magnitude, to determine the vertex, the problem is evidently indeterminate; for an unlimited number of different triangles may be constructed on the same base whose vertical angles are equal. But the vertices of all the triangles on the same side of the base will in this case be placed on the arc of a circle containing an angle equal to the given angle. Hence the *locus* will be two segments of circles containing an angle equal to the given angle, and constructed on opposite sides of the given base.

(21) The investigation of *loci* is of very extensive use in the solution of determinate problems. In cases where the determination of a point is required from certain data, by omitting any one of the data the point will have a *locus* which may be found by the remaining data. This being successively applied to two of the data, two loci will be found, the intersection of which will determine the point.

This may be illustrated by the examples already given. Let the base of a triangle be given in magnitude and position, and the area and vertical angle in magnitude, to determine the vertex. If we omit the vertical angle, the locus is the parallels already described. If we omit the area, the locus is the segments of the circle. The vertex being then at the same time on both loci must be at the intersection of the two loci, and will therefore be at the points where the parallels meet the circle. In general there will be, in the present case, four such points, and consequently four triangles, but these triangles will differ only in position, being equal as to their sides and angles.

The following propositions will illustrate the theory of Geometric loci.

PROPOSITION.

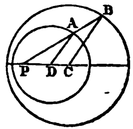
(22) *Given the base of a triangle, and the vertical angle, to determine the locus of the intersection of perpendiculars to the sides from the extremities of the base.*

These perpendiculars intersect at an angle supplemental to the vertical angle. The latter is given, and therefore the former. Hence the sought locus is a segment on the base containing an angle equal to the supplement of the vertical angle.

If the perpendiculars intersect below the base, which will happen when one of the base angles is obtuse, the locus is a segment on the opposite side of the base containing an angle equal to the vertical angle.

PROPOSITION.

(23) *A right line is drawn from a given point to the circumference of a circle given in magnitude and position, to determine the locus of a point which cuts it in a given ratio.*



If the given point be the centre of the given circle, it is evident that the locus sought is a concentric circle which cuts the radii of the given one in the given ratio.

If the point P be not at the centre of the given circle draw P C to the centre, and draw any line P B to the circumference. The ratio of P B : P A is given. Draw B C, and from A draw A D parallel to B C. Hence we have

$$PB : PA = BC : AD.$$

Hence the ratio B C : A D is given, and since B C is given in magnitude, A D is also given in magnitude. But because of the parallels,

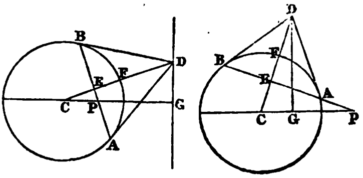
$$PA : PB = PD : PC.$$

Hence P C is cut in a given ratio at D, and therefore the point D is given. Since then the point D is given and the line D A is given in magnitude, the locus of the point A is a circle whose centre is D and radius D A. The demonstration is evident.

The same reasoning will apply whether the point P be within, or without, or on the circle.

PROPOSITION.

(24) *A circle is given in magnitude and position, and a chord passes through a given point, to find the locus of the intersection of tangents through the extremities of the chord.*



Let C B A be the circle, P the given point, A B any chord through it, and D the corresponding point of the locus. Draw C D, which will evidently bisect B A at right angles, and we have by the known properties of the circle C E : C F : C D. Hence the rectangle D C × C E is equal to the square of the radius C F. Draw D G perpendicular to C P produced, and the angles G and E being right, the quadrilateral D E P G may be circumscribed by a circle; therefore the rectangle D C × C E is equal to the rectangle G C × C P, and therefore the rectangle G C × C P is equal to the square of the radius. Hence the point G is independent of the point D, and a perpendicular from any point of the locus will meet C P produced at the same point D. Hence to construct the locus, find a third proportional to C P and the radius, and take C G equal to this third proportional, and through G draw a perpendicular to C G. This perpendicular will be the locus sought.

The nearer the given point P is to the centre, the more remote will be the locus G D, and when P coincides with the centre, C G will become infinite, so that in this case the locus may be considered a right line at an infinite distance.

There will be no difficulty in establishing the converse of this principle, *viz.* 'If tangents be drawn from each point in a given right line to a given circle, the chords joining the points of contact will all pass through a certain given point.'[*]

SECTION V.

Porisms.

(25) THE term *porism*[†] has been variously defined by Geometers. *Pappus* states, that Euclid wrote three books on porisms, (which have been lost,) but is so obscure and indistinct on the subject, that it is impossible merely from what he has stated to determine to what species of geometrical proposition the ancients applied this term.[‡] It is certain that it was sometimes used synonymously with *corollary;* thus Euclid, in his *Elements*, calls the corollaries of his propositions *πορισματα*. In an elaborate dissertation on the subject of *porisms*, in the *Transactions of the Royal Society of Edinburgh*, Playfair has, however, succeeded in giving the word a meaning more worthy of the importance which is evidently attached to this class of propositions. The porisms of Euclid are said to be '*collectio artificiosissima multarum rerum quæ spectant ad analysin difficiliorum et generalium problematum.*'

According to Playfair, a porism is 'a problem in which the data are so related to each other that it becomes indeterminate, and admits of numberless solutions.'

It is easily conceived that a problem which, in general, is determinate, will, when its data are submitted to certain conditions, become indeterminate. In such cases it becomes a *porism;* and it may be proposed in a porism to determine what condition or restriction will render a determinate problem indeterminate.

Thus, if it be required to draw a right line through a given point, subject to some given condition, the problem may be in general determinate; and it may be possible to draw but one such right line. But, on the other hand, such a position may be selected for the given point, as that every line passing through it will fulfil the given condition. When this position is assigned to the point, the problem becomes a porism. The following examples will render these observations more intelligible.

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[*] A numerous collection of Local problems will be seen in my treatise on *Algebraic Geometry*. The solutions there given are, however, by the Algebraical Analysis.

[†] From *πορίζω, I establish;* or, according to some, from *πόρος, a transition.*

[‡] Pappus defines a porism to be something between a theorem and problem, or that in which something is proposed to be investigated. Simson follows Pappus, and says, that a porism is a theorem or problem in which it is proposed to investigate or demonstrate something.

the given right line, it is given in position. Also G H and G I are given in magnitude, and therefore G D, which is a mean proportional between them, is given in magnitude, and therefore the point D is given in position.

(28) There is between local theorems and porisms a close analogy. In fact, every local theorem may be converted into a porism; but, on the contrary, every porism cannot be converted into a local theorem. In local propositions the indeterminate is always a point, the position of which is restricted, but not absolutely fixed by the given conditions. Such may always be expressed as a porism. But this class of propositions is more general than geometric loci; the indeterminate may be a *line*, the direction of which is not restricted by the conditions, but which is otherwise limited, as, for example, to pass through a given point, or to touch a given circle. It may also be a plane similarly restricted to pass through a given point, or to touch a given sphere. Instances of these have been given in (26).

Porisms, in common with geometric loci, take their rise from the conditions of a problem becoming indeterminate. This may happen in two ways. The number of conditions may not be sufficient, or among the given conditions there may exist some particular relation, by which some one or more of them may be deduced from the others. Thus, for the determination of a triangle, three conditions are necessary; and such a problem becomes manifestly indeterminate, if only two conditions be given. But even though three be given, the problem will still be indeterminate, if any one of the three can be inferred from the other two. For example, suppose the base of a triangle, the point where the perpendicular intersects it, and the difference of the squares of the sides be given, the problem to determine the triangle is indeterminate, because the difference of the squares of the sides is equal to the difference of the squares of the segments of the base, and may, therefore, be inferred from the base and the point of section.

The geometrical circumstances by which determinate problems in Geometry are converted into porismatic and local problems, are precisely similar to those under which the solution of an algebraical question becomes indeterminate. In such a question there should be as many equations as unknown quantities, and the problem is indeterminate evidently if there be less. But it may also be indeterminate, even if the number of equations be equal to that of the unknown quantities, and will be so when any one of the equations can be deduced from the others. It may in general be observed, both in geometrical and algebraical problems, that the number of *independent conditions* should be equal to the number of quantities sought, and should neither be more nor less. If they be more, the results may be inconsistent, and if they be less, the solution will be indeterminate.

In this case, it may be proved that the line must be a tangent to a circle, whose centre is the centre of mean distances, and whose radius is equal to the given line divided by the number of given points.

If the given points be not in the same plane, the porism may be made still more general: 'Given any number of points in space, to determine a plane passing among them, so that the sum of the perpendiculars from the points on one side shall exceed the sum of the perpendiculars from the points on the other side by a given line.'

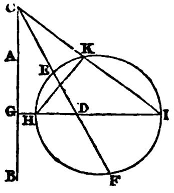
In this case the plane much touch a sphere whose centre is the centre of mean distances, and whose radius is the given line divided by the number of points.

If the sum of the perpendiculars on one side be equal to those on the other, the given line and the radius of the sphere vanish, and the sphere is reduced to its centre, *i. e.* the centre of mean distances. Hence, 'if a plane be drawn through the centre of mean distances, the sum of the perpendicular from the points on the one side is equal to the sum of the perpendiculars from the points on the other side.'

PROPOSITION.

(27.) *A circle and a straight line being given in position, a point may be found such that any right line drawn from it to the given line shall be a mean proportional between the parts of the same line, intercepted between the given right line and the circumference of the given circle.*

Let A B be the given right line, H K F the given circle, and D the sought point. Draw G D I perpendicular to A B through D, and also any other line C D F. Also join C I and draw H K.



The square of C D is equal to the rectangle C E × C F; but it is also equal to the squares of C G and G D, and the rectangle C E × C F is equal to the rectangle C K × C I. Hence the rectangle C K × C I is equal to the sum of the squares of C G and G D. The square of G D is equal to the rectangle G H × G I; therefore the rectangle G H × G I, together with the square of C G, is equal to the rectangle C K × C I. Also the square of C I is equal to the sum of the squares of C G and G I. But the square of C I is equal to the rectangle C K × C I, together with C I × K I, and the sum of the squares of C G and G I is equal to the square of C G, together with the rectangles G H × G I and G I × H I. Taking away from these equals the rectangle C K × C I, and its equivalent the rectangle G H × G I, together with the square of G C the remainders, the rectangles C I × I K and G I × I H are equal. Hence we have

$$GI : IC :: IK : IH.$$

Therefore, in the triangles C I G and H I K the angle I is common, and the sides which include it are proportional, and therefore the triangles are similar; but G is a right angle, and therefore H K I is a right angle, and therefore H I is a diameter. Since, then, H I passes through the centre of the given circle, and is perpendicular to A B the given right line, it is given in position. Also G H and G I are given in magnitude, and therefore G D, which is a mean proportional between them, is given in magnitude, and therefore the point D is given in position.

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SECTION VI.

Invention of Exercises.

I SHALL not attempt here to furnish the student with a collection of particular problems for the exercise of his ingenuity. To accomplish this would swell the bulk of the volume, and increase its price without any adequate advantage. I consider it better to offer such observations as may give a facility in the invention of geometrical exercises, and which, though applied to particular cases, may be very easily generalized.

In problems respecting the construction of a triangle, three independent data are necessary. Of the various lines and angles which have remarkable geometrical connection, or relation with a triangle, any three may be given to construct the triangle; and hence innumerable problems arise, which it would be almost as vain to attempt to enumerate as to solve. We shall, however, mention some of the most remarkable magnitudes thus connected with a triangle, and which most commonly form the data in such problems:

b, the base.
$\left.\begin{matrix}a\\c\end{matrix}\right\}$, the sides.
B, the vertical angle.
$\left.\begin{matrix}\mathrm{A}\\\mathrm{B}\end{matrix}\right\}$, the base angles opposite to a, b.
p, the altitude.
R, the radius of the circumscribed circle.
r, the radius of the inscribed circle.
b', the line from the vertex bisecting the base.
B$'$, the line drawn to the base bisecting the vertical angle.
S, the area.
P, the perimeter.

In general, any three of these may be assumed as data for the construction of the triangle. It is obvious, however, that relations may be instituted between innumerable other magnitudes and the triangle, so as to increase the number of data without limit.

Besides this, the data may be, not any of the single magnitudes above mentioned, but some magnitudes which depend on their combination in pairs by certain geometrical operations. Thus to determine the triangle, we may be given the base, the *sum* of the sides, and the *difference* of the base angles. In general, new data may be found by combining the above in pairs, by addition, or subtraction; or, if they be *lines*, by rectangles, squares, or ratios. By combining the two sides a, c alone, we obtain the following additional data:

$a \times c$, the rectangle under the sides.
$a + c$, their sum.
$a - c$, their difference.
$a : c$, their ratio.
$a^2 + c^2$, the sum of their squares.
$a^2 - c^2$, the difference of their squares.

Without pursuing this subject further, the student will perceive how he may invent for himself numberless problems respecting the construction of triangles.

In addition to the properties of right-angled triangles proved in the *Elements*, the student will find the following an useful exercise:

Let a, b be the sides, c the hypotenuse, s, s' the segments of the hypotenuse, p the perpendicular on the hypotenuse, and let the other letters retain the significations already assigned to them; the square of a line being signified by the number 2 placed above it.

1. $(a^2 + b^2) - (s^2 + s'^2) = 2\,p^2$.
2. $(c + r) = \frac{1}{2}\,\mathrm{P}$.
3. $r\,\mathrm{P} = c\,p$, (this is a property of every triangle.)
4. given r and c, to find the triangle.
5. given r and $a + b$..............
6. given r and a..............
7. $a\,b = r\,\mathrm{P} = 2\,r\,(c + r)$.
8. $p\,c = r\,\mathrm{P} = 2\,r\,(c + r)$.
9. $c\,(\frac{1}{2}\,p - r) = r^2$.
10. $\frac{1}{2}\,c\,p = r^2 + c\,r$.
11. $\mathrm{S} - r^2 = c\,r$.
12. $(c + p)^2 - (a + b)^2 = p^2$.
13. $\{a + b + c + p\} \times \{(c + p) - (a + b)\} = p^2$.
14. The triangle formed by $a + b$, p and $c + p$ is right-angled.
15. $(c - p)^2 - (a - b)^2 = p^2$.
16. $\{(c - p) + (a - b)\} \times \{(c - p) - (a - b)\} = p^2$.
17. The triangle formed by $(c - p)$, $(a - b)$, and p, is right-angled.

A numerous class of problems have for their object to draw a right line from a given point intersecting a circle under given conditions. The conditions by which the secant may be restricted in this case are infinitely various. They may have reference to any of the three intercepts between the given point and the two points of intersection with the circle, and any given relation may be required to subsist between any two of these intercepts. Again, in addition to a circle, a right line may be intersected by the line from the given point. This will still further increase the number of relations which may be selected as data for the problem; or, instead of a right line, a second circle may be given.

An extensive class of problems respecting *maxima* and *minima* arise out of these considerations; for, in most instances, some of the intercepted parts admit of limiting magnitudes.

Figures inscribed within and circumscribed around each other under given conditions, form also a large class of exercises. In these the considerations of maxima and minima may be introduced with advantage.

No. II.

ON THE THEORY OF PARALLEL LINES.

THE theory of parallel lines has always been considered as the reproach of Geometry. The beautiful chain of reasoning by which the truths of this science are connected here wants a link, and we are reluctantly compelled to assume as an *axiom* what ought to be matter of demonstration. The most eminent geometers, ancient and modern, have attempted without success to remove this defect; and after the labours of the learned for 2000 years have failed to improve or supersede it, Euclid's theory of parallels maintains its superiority. We shall here endeavour to explain the nature of the difficulty which attends this investigation, and shall give some account of the theories which have been proposed as improvements on, or substitutes for, that of Euclid.

Of the properties by which two right lines described upon the same plane are related, there are several which characterise two parallel lines and distinguish them from lines which intersect. If any one of such properties be assumed as the definition of parallel lines, all the others should flow demonstratively from it. As far therefore as the strict principles of logic are concerned, it is a matter of indifference which of the properties be taken as the definition. In the choice of a definition, however, we should be directed also by other circumstances. That property is obviously to be preferred from which all the others follow with greatest ease and clearness. It is also manifest that, *cæteris paribus*, that property should be selected which is most conformable to the commonly received notion of the thing defined. These circumstances should be attended to in every definition, and the exertion of considerable skill is necessary almost in every case. But in the selection of a definition for parallel lines there is a difficulty of another kind. It has been found, that whatever property of parallels be selected as the basis of their definition, the deduction of all the other properties from it was impracticable. Under these circumstances the only expedient which presents itself, is to assume, besides the property selected for the definition, another property as an axiom. This is what every mathematician who has attempted to institute a theory of parallel lines has done. Some, it is true, have professed to dispense with an axiom, and to derive all the properties directly from their definition. But these, with a single exception, which we shall mention hereafter, have fallen into an illogicism inexcusable in geometers. We find invariably a *petitio principii*, either incorporated in their definition, or lurking in some complicated demonstration. A rigorous dissection of the reasoning never fails to lay bare the sophism.

Of the pretensions of those who avowedly assume an axiom it is easy to judge. When Euclid's axiom is removed from the very disadvantageous position which it has hitherto maintained, put in its natural place, and the terms in which it is expressed somewhat changed, I think it will be acknowledged that no proposition which has ever yet been offered as a substitute for it is so nearly self-evident. But it is not alone in the degree of self-evidence of his axiom, if we be permitted the phrase, that Euclid's theory of parallels is superior to those theories which are founded on other axioms. The superior simplicity of the structure which he has raised upon it is still more conspicuous. When you have once admitted Euclid's axiom, all his theorems flow from that and his definition, as the most simple and obvious inferences. In other theories, after conceding an axiom much further removed from self-evidence than Euclid's, a labyrinth of complicated and indirect demonstration remains to be threaded, requiring much subtlety and attention to be assured that error and fallacy do not lurk in its mazes.

Euclid selects for his definition that property in virtue of which parallel lines, though indefinitely produced, can never intersect. This is, perhaps, the most ordinary idea of parallelism. Almost every other property of parallels requires some consideration before an uninstructed mind assents to it; but the possibility of two right lines intersecting, is repugnant to every notion of parallelism.

When the possible existence of the subject of a definition is not self-evident, or presumed and declared to be so, it ought to be proved so. This is the case with Euclid's definition of parallels. How, it may be asked, does it appear that two right lines *can* be drawn upon the same plane so as never to intersect though infinitely produced? Euclid meets this objection in his 27th proposition, where he shows that if two lines be inclined at equal alternate angles to a third, the supposed possibility of their intersection would lead to a manifest contradiction. Thus it appears, that through a given point *one* right line *at least* may always be drawn parallel to a given right line. But it still remains to be shown, that *no more than one* parallel can be drawn through the same point to the same right line. And here the chain of proof is broken. Euclid was unable to demonstrate, that every other line except that which makes the alternate angles equal will necessarily intersect the given right line if both be sufficiently produced. He accordingly found himself compelled to place the deficient link among his axioms.

I have always conceived that the objections which have been urged against the twelfth axiom have arisen from the place assigned to it in the *Elements*. Standing as it does on the threshhold of the first book, it is contemplated without reference to those propositions with which it is or ought to be immediately connected, which prepare the mind of the student for its reception, and which seem almost indispensable to render the very terms in which it is expressed intelligible. This axiom, as we have stated, is only assumed in the demonstration of the 29th proposition. In the 27th it is proved that if B E F (see fig. Prop. XXVII. Book I.) and D E F be together equal to two right angles E, B is parallel to F D. The axiom declares that no other right line through E can be parallel to F D; for it is plain, that any other line must make with E F an angle which together with D F E is less or greater than two right angles. The twelfth axiom may therefore be expressed in any of the following ways:

'Two diverging right lines cannot be both parallel to the same right line.'

'If a right line intersect one of two parallel right lines it must also intersect the other.'

'Only one right line can be drawn through a given point parallel to a given right line.'

The axiom expressed by the first of these ways appears to me to be as unobjectionable, as any of those which have been received without dispute. *Playfair*, in his edition of Euclid, and before him *Ludlam*, expresses the axiom nearly in this way, but he does not seem to be aware that it is the same axiom as that of Euclid; for he says in his preface that a *new axiom* is introduced in place of the 12th. In fact, it is the same axiom otherwise expressed.

Proclus objects to Euclid's axiom, that it is less entitled to be considered self-evident, because the converse of it (XXVIII. Book I.) confessedly requires proof. Are we hence to infer that Proclus considered no proposition to be self-evident, unless its converse be also self-evident? If this were admitted, I fear it would be fatal to some of the axioms which Proclus himself, in common with the rest of the world, have received. Neither the second nor third of Euclid's axioms respecting equal magnitudes could by this rule be admitted as such; for, so far from their converses being self-evident, *they are not even true*. Proclus would seem to have attributed to an axiom the quality of a definition, which must always be what logicians call a reciprocal proposition; but axioms are universal affirmative propositions not necessarily reciprocal, and therefore notoriously not logically convertible.

I shall not attempt to go into the particular details of the various theories of parallels which have been proposed, nor even to enumerate them. I shall, however, mention some of those which, from the eminence of their authors, if from nothing else, must command attention.

Clavius rejects Euclid's axiom, and proposes the following as a substitute for it. 'A line drawn upon a plane from every point of which perpendiculars on a right line in the same plane are equal, is itself a right line.' From this proposition assumed as an axiom by a most complex and embarrassing process, he shows that the properties of parallels may be deduced.

Wolfius, *Boscovich*, and *Thomas Simson*, change the definition of parallels and substitute for it the following: 'Two right lines are parallel when perpendiculars from every point in one upon the other are equal.' This definition is sophistical, and really *assumes* the axiom of Clavius. *D'Alembert*, with more acuteness, proposes to define parallels to be 'right lines, one of which has two points equally distant from the other.' But the difficulty of demonstrating that the other points are equally distant, he acknowledges still to remain undiminished.'

Thomas Simson, in his second edition, proposes the following axiom: 'If perpendiculars from two points of a straight line upon another straight line in the same plane be unequal, the two lines will meet if indefinitely produced on the side of the least distance.'

Robert Simson proposes the following axiom: 'That a straight line cannot first come nearer to another straight line, and then go further from it, without meeting it.' The meaning of which is, that if from any three points in one right line perpendiculars be drawn to another, they will either be equal, or the intermediate perpendicular will be greater than one of the extreme perpendiculars and less than the other.

Varignon, *Bezout*, and others, propose the following definition as a substitute for Euclid's: 'Two right lines in the same plane are parallel when they are equally inclined in the same direction to a third right line.' By this either of two things is meant; that the parallels are equally inclined to *one particular line* intersecting them, or that they are equally inclined to *every line* intersecting them. If it be taken in the former sense it is insufficient, and if in the latter it is sophistical. If they be equally inclined to *one particular line*, it remains to be proved that they are also equally inclined to every other intersecting line, which in fact is, and always has been, the whole difficulty of the question. If it be meant that they are equally inclined to every intersecting line, it is a sophism, in which a theorem is presented in the garb of a definition.

Professor *Leslie* retains Euclid's definition in substance, though somewhat changed in expression. By a singular oversight, he has appended to the enunciation of Euclid's 29th proposition, demonstrations of his 27th and 28th, thus leaving the 29th without a proof.

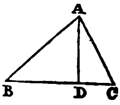
From these specimens the success of the various attempts to mend Euclid's theory of parallels may be estimated. We must, however, make honourable exception of *Legendre*. His system, considered as a part of the most elementary mathematical treatise, is certainly liable to objection; but the objection is of a very different character from those which lie against all the others. *Legendre* assumes no axiom, makes no change in Euclid's definition, and admits no latent assumption or other fallacy in his reasoning. After his first theorem, which is the 32d proposition of Euclid, his demonstrations are as simple as those of Euclid, and even attended with superior advantages. To comprehend his first demonstration, however, requires a greater familiarity with the language and principles of analysis than students commencing the elements of geometry generally possess. It is to be regretted that a system must therefore be abandoned which is in other respects incomparably the best, and, indeed, the only one which is altogether free from objections on the score of validity.

Legendre has given several demonstrations of the 32d proposition; they are, however, all subject more or less to the objection just mentioned. We shall subjoin that which is most remarkable for its brevity, its elegance, and the importance and extent of its applications.

By superposition, it can be shown immediately, and without any preliminary propositions, that *two triangles are equal when they have two angles and an interjacent side in each equal.* Let us call this side p, the two adjacent angles A and B, the third angle C. This third angle C, therefore, is entirely determined, when the angles A and B, with the side p, are known; for if several different angles C might correspond to the three given magnitudes A, B, p, there would be several different triangles, each having two angles and the interjacent side equal, which is impossible; hence the angle C must be a determinate function of the three quantities A, B, p, which we shall express thus, $C = \phi : (A, B, p)$.

Let the right angle be equal to unity, then the angles A, B, C will be numbers included between 0 and 2; and since $C = \phi : (A, B, p)$, the line p cannot enter into the function ϕ. For we have already seen that C must be entirely determined by the given quantities A, B, p alone, without any other line or angle whatever. But the line p is heterogeneous with the numbers A, B, C; and if there existed any equation between A, B, C, p, the value of p might be found from it in terms of A, B, C; whence it would follow, that p is equal to a number; which is absurd: hence p cannot enter into the function ϕ, and we have simply $C = \phi : (A, B)$.*

This formula already proves, that if two angles of one triangle are equal to two angles of another, the third angle of the former must also be equal to the third of the latter; and this granted, it is easy to arrive at the theorem we have in view.



First, let A B C be a triangle right-angled at A; from the point A draw A D perpendicular to the hypotenuse. The angles B and D of the triangle A B D are equal to the angles B and A of the triangle B A C; hence, from what has just been proved, the third angle B A D is equal to the third C. For a like reason, the angle $DAC = B$, hence $BAD + DAC$, or $BAC = B + C$; but the angle B A C is right; hence *the two acute angles of a right-angled triangle are together equal to a right angle.*

Now, let B A C be any triangle, and B C a side of it not less than either of the other sides; if from the opposite angle A the perpendicular A D is let fall on B C, this perpendicular will fall within the triangle A B C, and divide it into two right-angled triangles B A D, D A C. But in the right-angled triangle B A D, the two angles B A D,

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* Against this demonstration it has been objected, that if it were applied word for word to spherical triangles, we should find that two angles being known, are sufficient to determine the third, which is not the case in that species of triangles. The answer is, that in spherical triangles there exists one element more than in plane triangles, the radius of the sphere, namely, which must not be omitted in our reasoning. Let r be the radius; instead of $C = \varphi\,(A, B, p)$, we shall now have $C = \varphi\,(A, B, p, r)$, or by the law of homogeneity, simply $C = \varphi \left(A, B, \frac{p}{r}\right)$. But since the ratio $\frac{p}{r}$ is a number, as well as A, B, C, there is nothing to hinder $\frac{p}{r}$ from entering the function φ, and, consequently, we have no right to infer from it, that $C = \varphi(A, B)$.

A B D are together equal to a right angle; in the right-angled triangle D A C, the two D A C, A C D are also equal to a right angle; hence all the four taken together, or, which amounts to the same thing, all the three, B A C, A B C, A C B, are together equal to two right angles; hence *in every triangle, the sum of its three angles is equal to two right angles.*

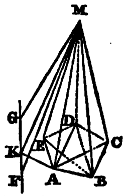
It thus appears, that the theorem in question does not depend, when considered *à priori*, upon any series of propositions, but may be deduced immediately from the principle of homogeneity; a principle which must display itself in a relation subsisting between all quantities of whatever sort. Let us continue the investigation, and show that, from the same source, the other fundamental theorems of geometry may likewise be derived.

Retaining the same denominations as above, let us further call the side opposite the angle A by the name of m, and the side opposite B by that of n. The quantity m must be entirely determined by the quantities A, B, p alone; hence m is a function of A, B, p, and $\frac{m}{p}$ is one also; so that we may put $\frac{m}{p} = \psi : (\mathrm{A}, \mathrm{B}, p)$. But $\frac{m}{p}$ is a number as well as A and B; hence the function ψ cannot contain the line p, and we shall have simply $\frac{m}{p} = \psi : (\mathrm{A}, \mathrm{B})$, or $m = p\,\psi : (\mathrm{A}, \mathrm{B})$. Hence, also, in like manner, $n = p\,\psi\ (\mathrm{B\,A})$.

Now, let another triangle be formed with the same angles A, B, C, and with sides m', n', p', respectively opposite to them. Since A and B are not changed, we shall still, in this new triangle, have $m' = p'\psi : (\mathrm{A}, \mathrm{B})$, and $n' = p'\psi : (\mathrm{A}, \mathrm{B})$. Hence $m : m' :: n : n' :: p : p'$. Hence *in equiangular triangles, the sides opposite the equal angles are proportional.*

The proposition concerning the square of the hypotenuse, is a consequence of that concerning equiangular triangles. Here then are three fundamental propositions of geometry,—that concerning the three angles of a triangle, that concerning equiangular triangles, and that concerning the square of the hypotenuse, which may be very simply and directly deduced from the consideration of functions. In the same way, the propositions relating to similar figures and similar solids may be demonstrated with great ease.

Let A B C D E be any polygon. Having taken any side A B, upon A B as a base, form as many triangles A B C, A B D, &c. as there are angles C, D, E, &c. lying out of it. Put the base A B $= p$; let A and B represent the two angles of the triangle A B C, which are adjacent to the side A B; A′ and B′ the two angles of the triangle A B D, which are adjacent to the same side A B, and so on. The figure A B C D E will be entirely determined, if the side p with the angles A, B, A′, B′, A″, B″, &c. are known, and the number of data will in all amount to $2n - 3$, n being the number of the polygon's sides. This being granted, any



side or line x, any how draw in the polygon, and from the data alone which serve to determine this polygon, will be a function of those given quantities; and since $\frac{x}{p}$ must be a number, we may suppose $\frac{x}{p} = \psi : (A, B, A', B', \&c.)$ or $x = p\,\psi\,(A, B, A'\,B', \&c.)$, and the function ψ will not contain p. If with the same angles, and another side p', a second polygon be formed, the line x' corresponding or homologous to x will have for its value $x' = p'\,\psi : (A, B, A'\,B', \&c.)$; hence $x : x' :: p : p'$. Figures thus constructed might be defined as *similar figures*; hence *in similar figures the homologous lines are proportional.* Thus, not only the homologous sides and the homologous diagonals, but also any lines terminating the same way in the two figures, are to each other as any other two homologous lines whatever.

Let us name the surface of the first polygon S; that surface is homogeneous with the square p^2; hence $\frac{S}{p^2}$ must be a number, containing nothing but the angles A, B, A′, B′, &c.; so that we shall have $S = p^2\phi : (A, B, A', B', \&c.)$; for the same reason, S′ being the surface of the second polygon, we shall have $S' = p'^2\phi : (A, B, A', B', \&c.)$ Hence $S : S' :: p^2 : p'^2$; hence *the surfaces of similar figures are to each other as the squares of their homologous sides.*

Let us now proceed to polyedrons. We may take it for granted, that a face is determined by means of a given side p, and of the several given angles A, B, C, &c. Next, the vertices of the solid angles which lie out of this face, will be determined each by means of three given quantities, which may be regarded as so many angles; so that the whole determination of the polyedron depends on one side, p, and several angles A, B, C, &c. the number of which varies according to the nature of the polyedron. This being granted, a line which joins to no vertices, or more generally, any line x drawn in a determinate manner in the polyedron, and from the data alone which serve to construct it, will be a function of the given quantities p, A, B, C, &c.; and since $\frac{x}{p}$ must be a number, the function equal to $\frac{x}{p}$ will contain nothing but the angles A, B, C, &c., and we may put $x = p\,\phi : (A, B, C, \&c.)$ The surface of the solid is homogeneous to p^2; hence that surface may be represented by $p^2\psi : (A, B, C, \&c.)$: its solidity is homogeneous with p^3, and may be represented by $p^3\,\Pi : (A, B, C, \&c.)$, the functions designated by ψ and Π being independent of p.

Suppose a second solid to be formed with the same angle A, B, C, &c., and a side p' different from p; and that the solids so formed are called *similar solids.* The line which in the former solid was $p\,\phi$, (A, B, C, &c.), or simply $p\,\phi$, will in this new solid become $p'\,\phi$; the surface which was $p^2\,\psi$ in the one, will now become $p'^2\,\psi$ in the other; and, lastly, the solidity which was $p^3\,\Pi$ in the one, will now become $p'^3\,\Pi$ in the other. Hence, first, *in similar solids, the homologous lines* *are proportional;* secondly, *their surfaces are as the squares of the homologous sides;* thirdly, *their solidities are as the cubes of those same sides.*

The same principles are easily applicable to the circle. Let c be the circumference, and s the surface of the circle whose radius is r; since there cannot be two unequal circles with the same radius, the quantities $\frac{c}{r}$ and $\frac{s}{r^2}$ must be determinate functions of r: but as these quantities are numbers, the expression of them cannot contain r; and thus we shall have $\frac{c}{r} = \alpha$, and $\frac{s}{r^2} = \beta$, α and β being constant numbers. Let c' be the circumference, and s' the surface of another circle whose radius is r'; we shall, as before, have $\frac{c'}{r'} = \alpha$, and $\frac{s'}{r'^2} = \beta$. Hence $c : c' :: r : r'$, and $s : s' :: r^2 : r'^2$; hence *the circumferences of circles are to each other as their radii, and the surfaces are as the squares of those radii.*

Let us now examine a sector whose radius is r. A being the angle at the centre, let x be the arc which terminates the sector, and y the surface of that sector. Since the sector is entirely determined when r and A are known, x and y must be determinate functions of r and A; hence $\frac{x}{r}$ and $\frac{y}{r^2}$ are also similar functions. But $\frac{x}{r}$ is a number, as well as $\frac{y}{r^2}$; hence those quantities cannot contain r, and are simply functions of A; so that we have $\frac{x}{r} = \phi : \mathrm{A}$, and $\frac{y}{r^2} = \psi : \mathrm{A}$. Let x' and y' be the arc, and the surface of another sector, whose angle is A, and radius r'; we shall call those two sectors *similar:* and since the angle A is the same in both, we shall have $\frac{x'}{r'} = \phi : \mathrm{A}$, and $\frac{y'}{r'^2} = \psi : \mathrm{A}$. Hence $x : x' :: r : r'$, and $y : y' :: r^2 : r'^2$; hence *similar arcs,* or *the arcs of similar sectors, are to each other as their radii; and the sectors themselves are as the squares of the radii.*

By the same method we could evidently show, that spheres are as the cubes of their radii.

In all this we have supposed that surfaces are measured by the product of two lines, and solids by the product of three; a truth which it is easy to demonstrate by analysis, in like manner. Let us examine a rectangle, whose sides are p and q; its surface, which must be a function of p and q, we shall represent by $\phi : (p, q)$. If we examine another rectangle, whose dimensions are $p + p'$ and q, this rectangle is evidently composed of two others, of one having p and q for its dimensions, of another having p' and q; so that we may put $\phi : (p + p', q) = \phi : (p, q) + \phi : (p', q)$. Let $p' = p$; we shall have $\phi\,(2p, q) = 2\phi c\,(p, q)$. Let $p' = 2p$; we shall have $\phi\,(3p, q) = \phi\,(p, q)$

$+\phi(2p,q)=3\phi(p,q)$. Let $p'=3p$; we shall have $\phi(4p,q) = \phi(p,q)+\phi(3p,q)=4\phi(p,q)$. Hence, generally, if k is any whole number, we shall have $\phi(kp,q)=k\phi(p,q)$, or $\frac{\phi(p,q)}{p} = \frac{\phi(kp,q)}{kp}$; from which it follows that $\frac{\phi(p,q)}{p}$ is such a function of p as not to be changed by substituting in place of p any multiple of it kp. Hence this function is independent of p, and cannot include any thing except q. But for the same reason $\frac{\phi(p,q)}{q}$ must be independent of q; hence $\frac{\phi(p,q)}{pq}$ includes neither p nor q, and must therefore be limited to a constant quantity a. Hence we shall have $\phi(p,q)=apq$; and as there is nothing to prevent us from taking $a=1$, we shall have $\phi(p,q)=pq$; thus the surface of a rectangle is equal to the product of its two dimensions.

In the very same manner we could show, that the solidity of a right-angled parallelopipedon, whose dimensions are p, q, r, is equal to the product pqr of its three dimensions.

We may observe, in conclusion, that the docrine of functions, which thus affords a very simple demonstration of the fundamental propositions of geometry, has already been employed with success in demonstrating the fundamental principles of Mechanics. See the *Memoirs of Turin*, vol. ii.

THE END.

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